An Introduction to Control Theory

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Topics

- What? Why?
- How do I represent a system?
- Laplace Transform
- Complex Numbers
- Frequency Analysis
- DCA Example

Text Book

- An Introduction to Control Systems
  - Kevin Warwick, World Scientific,
  - ISBN 981-02-2597-0
  - £18.05 from Amazon

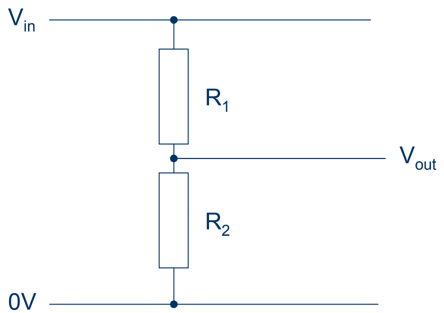
Control Theory

- "Control theory is an interdisciplinary branch of engineering and mathematics, that deals with the behavior of dynamical systems."
  - *Wikipedia*

•The science of:
  •What will happen to a system if I do this?
  •How do I get what I want out of this system?
  •How does this system work?

Representation of Systems

- A system: The potential divider



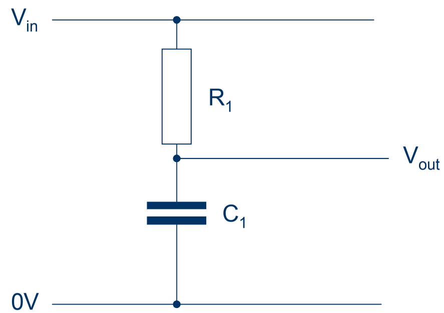
From Ohm's Law…

$$V_{out} = V_{in}\frac{R_2}{R_1 + R_2}$$

$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2}$$

Representation of Systems

- A system…



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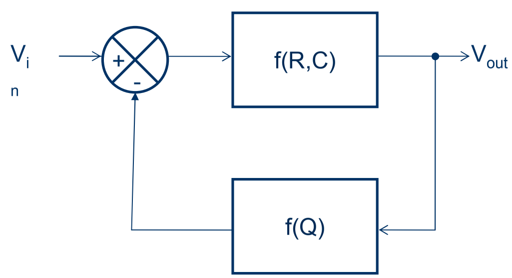
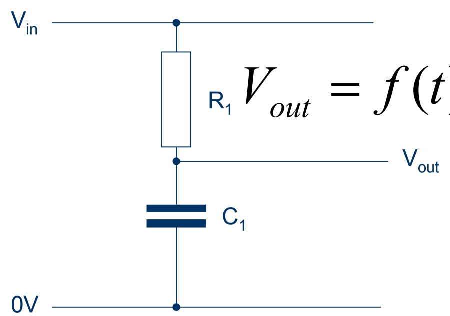
Feedback



Domains

- The time domain



$$V_{out} = f(t)$$

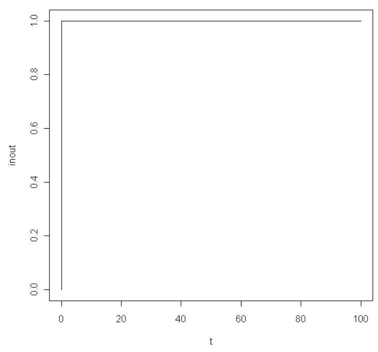
A Step Input

$$V_{out} = V_{in}\left(1 - e^{-\frac{t}{R_1C_1}}\right)$$

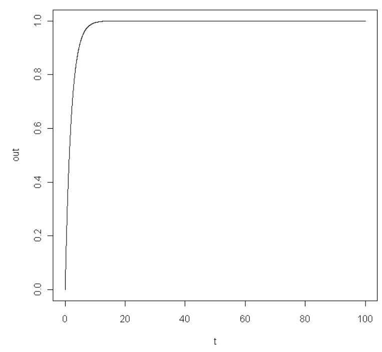
A Pulse Input

$$V_{out} = V_{in}\left(\frac{e^{-\frac{t}{R_1C_1}}}{R_1C_1}\right)$$

Results – Step Input (R=100,C=0.02)



Input



Output

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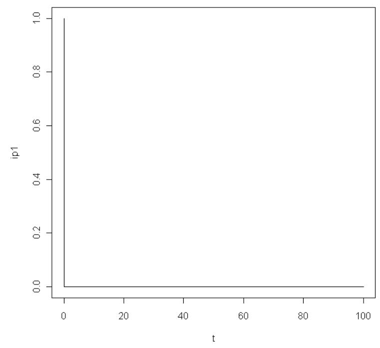
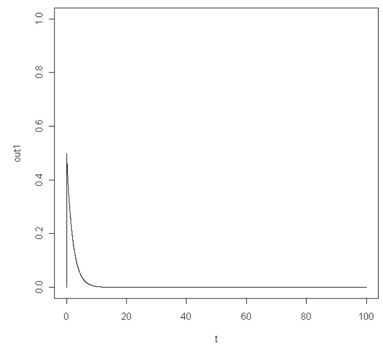
Results – Pulse Input (R=100,C=0.02)



Input



Output

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Domains

- We need a new function for every input in the time domain.
- Time-consuming, memory intensive.
- One system – one representation regardless of input!

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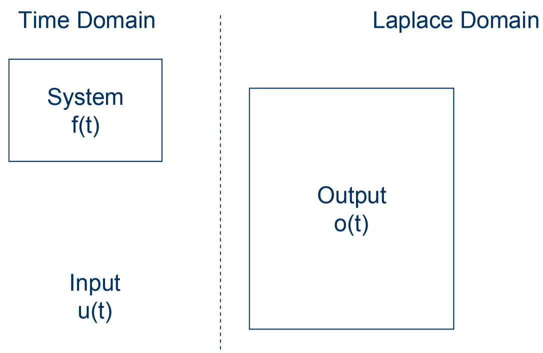
A Summary of Laplace Transforms



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Domains

- We can represent the system in the "Laplace Domain"

$$F(s) = L\{f(t)\}$$

- To get the output of a system, simply multiply the representation by the input!

$$O(s) = F(s) * I(s)$$

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Converting into the Laplace Domain

- The transform itself is ugly and unpleasant!

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t).dt$$

- Two alternative methods…
  - Rules exist to directly write Laplace transfer functions for many common system types
    - Electrical
    - Mechanical
    - Digital control
  - Look up tables to go from the time domain

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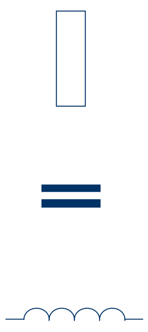
Electrical Systems



Resistor	R
Capacitor	$\frac{1}{Cs}$
Inductor	Ls

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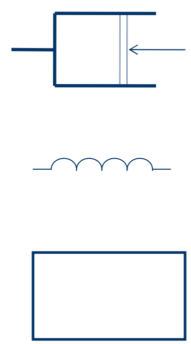
Mechanical Systems



Dashpot	K
Spring	$\frac{K}{s}$
Mass	Ms

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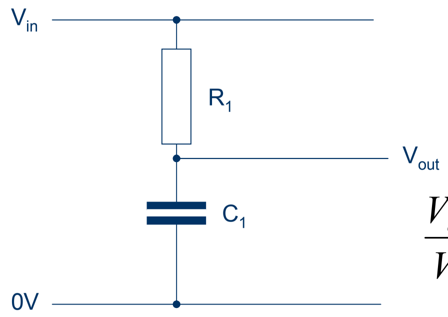
Lookup Tables

	Time domain	Laplace domain
Single Impulse	$\partial(t)$	1
Unit Step	1	s^{-1}
Unit Ramp	t	s^{-2}
Exponential decay	$e^{-\alpha t}$	$(s+\alpha)^{-1}$
Sine Wave	$Sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Inverse Exponential	$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$

Our Circuit Example



$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1+R_2}$$

$$\frac{V_{out}}{V_{in}} = H(s) = \frac{\frac{1}{C_1 s}}{\frac{1}{C_1 s}+R_1}$$

Circuit Example

$$H(s) = \frac{\frac{1}{C_1 s}}{\frac{1}{C_1 s}+R_1}$$

Rearranging…

$$H(s) = \frac{\frac{1}{R_1 C_1}}{\frac{1}{R_1 C_1}+s}$$

Lets multiply this by an input – try a unit step

$$U(s)*H(s) = \frac{1}{s}*\frac{\frac{1}{R_1 C_1}}{\frac{1}{R_1 C_1}+s} = \frac{\alpha}{s(s+\alpha)}$$

Where $\alpha = \frac{1}{R_1 C_1}$

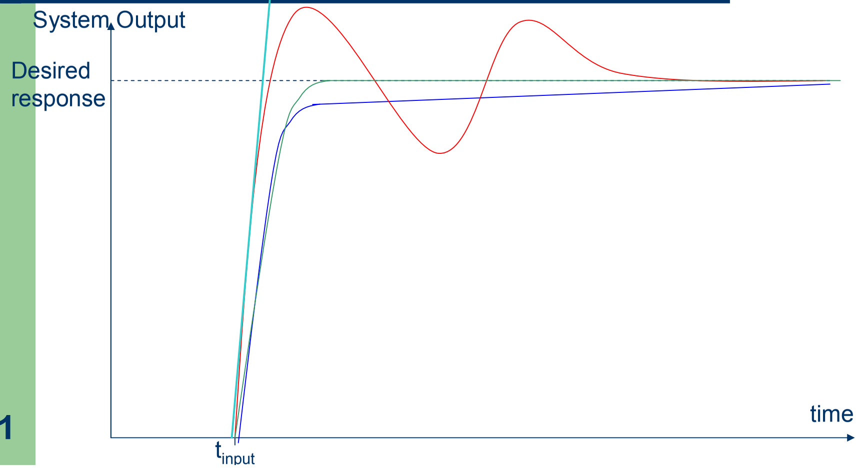
Circuit Example

Converting back to the time domain…

$$\frac{\alpha}{s(s+\alpha)} \Rightarrow 1-e^{-\alpha t} \qquad \alpha = \frac{1}{R_1 C_1}$$

$$1-e^{-\frac{1}{R_1 C_1}t}$$

Common Step Responses



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A Mathematical Aside

- Square roots

$$\sqrt{xy} = \sqrt{x} \times \sqrt{y}$$

$$\sqrt{36} = \sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$$

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A Mathematical Aside

$4 \times 4 = 4^2 =$ 16

$(-4) \times (-4) = (-4)^2 =$ 16

$\sqrt{16} =$ ±4

$\sqrt{-16} =$ $\sqrt{-1} \times \sqrt{16} = i \times 4$

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Complex Numbers

- We often talk about the set of Real numbers
- There are another set of numbers called "Imaginary Numbers" – the square roots of negative numbers!
- A number that has a real and an imaginary part is called a complex number

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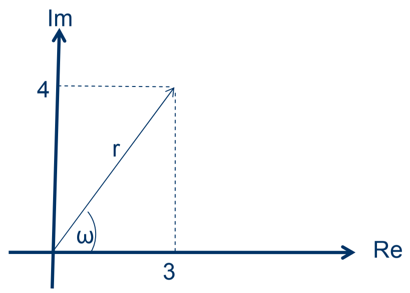
Complex Numbers



$$3 + 4i$$

$$r = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = 5$$

$$\omega = Tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = 53.13° = 1.42 rads$$

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More Representations

$$3 + 4i$$

$$r(Cos(\omega) + iSin(\omega))$$

$$re^{-i\omega}$$

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Why Complex Numbers?

- Demonstration

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Frequency Analysis

- The science of asking how a system responds to difference frequencies of signal
- WHY!!??

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Fourier

- Any signal can be modelled as a sum of sine waves at different frequencies and different magnitudes
- By knowing how a system reacts to all frequencies we know how it will react to all signals

The Frequency Domain

- Another domain!
- It's very easy to move from the Laplace domain into the Frequency domain

$$s = i\omega$$

Back to the Circuit

$$H(s) = \frac{\frac{1}{R_1C_1}}{\frac{1}{R_1C_1} + s} = \frac{\alpha}{\alpha + s}$$

$$H(i\omega) = \frac{\alpha}{\alpha + i\omega} = \frac{\alpha}{\alpha + i\omega} \times \frac{\alpha - i\omega}{\alpha - i\omega}$$

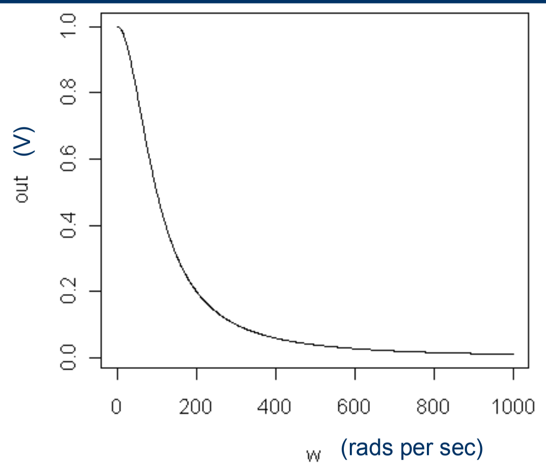
$$= \frac{\alpha^2 - i\alpha\omega}{\alpha^2 + i\alpha\omega - i\alpha\omega + \omega^2} = \frac{\alpha^2}{\alpha^2 + \omega^2} - i\left(\frac{\alpha\omega}{\alpha^2 + \omega^2}\right)$$

Back to the Circuit

$$H(\omega) = \frac{\alpha^2}{\alpha^2 + \omega^2} - i\left(\frac{\alpha\omega}{\alpha^2 + \omega^2}\right)$$

$$\mathrm{Re}\{H(\omega)\} = \frac{\alpha^2}{\alpha^2 + \omega^2}$$

Response (alpha=100)



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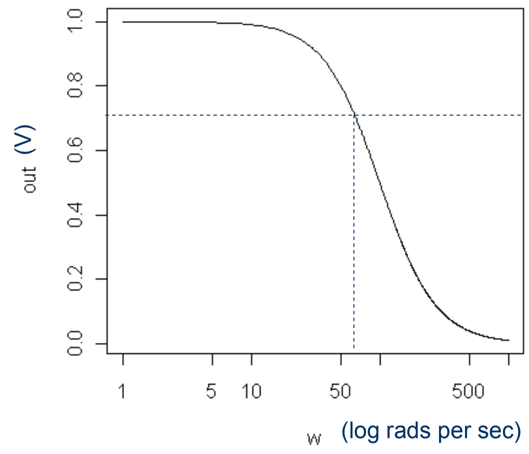
Response (alpha=100)



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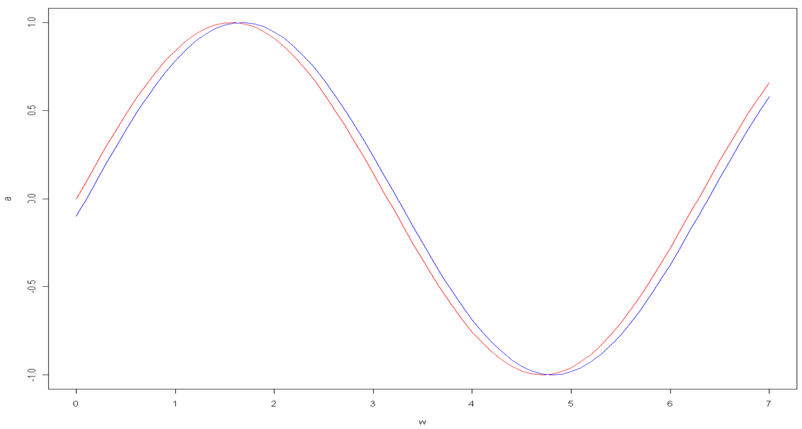
Phase Shift



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Phase Shifts

- The angle of the complex gain is the phase shift of the system
- The circuit…

$$Tan^{-1}\frac{\mathrm{Im}}{\mathrm{Re}} = Tan^{-1}\frac{\left(\frac{\alpha\omega}{\alpha^2+\omega^2}\right)}{\frac{\alpha^2}{\alpha^2+\omega^2}} = Tan^{-1}\frac{\omega}{\alpha}$$

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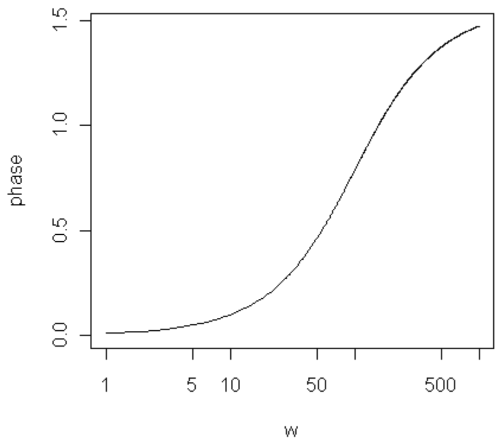
Phase



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Dendritic Cell Algorithm Modelling

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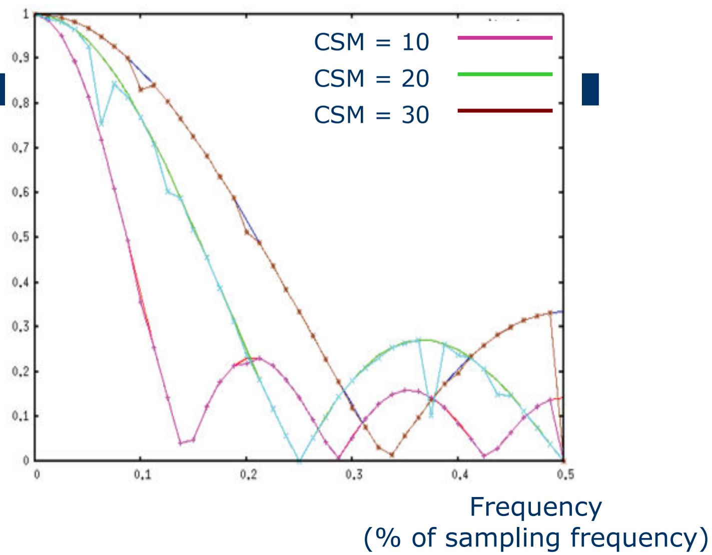
Dendritic Cell Modelling



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