IMA Seminar

An Overview of Fuzzy Systems

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Classical/Crisp Logic

- Origins in Ancient Greece
  - Aristotle
  - Plato
- Two truth values: true, false
- Connectives: not, and, or
- Aristotle saw weaknesses
  - future events: ?
- Zeno’s paradoxes
Fuzzy Logic

- Fuzzy logic introduced by Lotfi Zadeh in 1965
  - an extension of multi-valued logics
  - allows the representation of varieties of uncertainty
- It allows for reasoning with linguistic variables
  
  old people are often long sighted
  long sighted people need reading glasses
  THEREFORE
  old people often need reading glasses
- Fuzzy logic is currently probably the most popular form of uncertainty representation

Diagramatically

- The set of real numbers greater than 4, \( \mu_A(x) = 1 \) iff \( x \geq 4 \), may be illustrated as
Tall People

- But what about the set of tall people? Say, $\mu_{\text{tall}}(x) = 1$ iff $x \geq 66$ (inches, i.e. 5'6”)???

Fuzzy Tall People

- Let’s modify the sharp (crisp) cut-off for the set of tall people into a smooth transition, e.g.
Examples

- Examples of further fuzzy sets for height

![Graph showing examples of fuzzy sets for height.]

Multiple Memberships

- Importantly, a given element or value can belong to multiple fuzzy sets with differing memberships

![Graph showing multiple memberships for height.]

\[ \mu_{\text{tall}}(5'11") = 0.8 \]
\[ \mu_{\text{medium}}(5'11") = 0.2 \]
Piecewise Linear

- Left shoulder, triangular, trapezoid, right slope membership functions

Gaussians

$$\mu(x) = e^{-\frac{(x-c)^2}{2\sigma^2}}$$
Informal Definition

- A *linguistic variable* is a collection of fuzzy sets representing linguistic terms of a concept.

![Diagram of height categories](image)

Terms

- The number and shape of terms may be application dependent.

![Diagram of age categories](image)
Meaning of Grades

• What do membership grades mean?

• Consider
  – the membership grade of Jon in set of tall people is 0.9

• Likelihood view
  – 90% of the population would describe Jon as tall

• Random set view
  – 90% of the population described ‘tall’ as an interval containing Jon’s height

• Typicality view
  – Jon’s height is 90% along the scale of tallness
Fuzzy Sets and Probabilities

• Fuzzy memberships are not probabilities
  – there is no probability involved in a person’s height
  – memberships are better interpreted as compatibilities

• Consider you are given a bottle of liquid
  – scenario A: the liquid is drinkable with probability 0.9
  – scenario B: the liquid is drinkable with membership 0.9

• What is the difference?
  – scenario A: there is 0.1 chance that the bottle is filled with undrinkable liquid (poison?): potentially risky!
  – scenario B: the liquid is basically drinkable, but maybe not as nice as (say) beer with drinkability 1.0!

Complement

• The fuzzy complement, $\tilde{A}$, of fuzzy set $A$ is given by $\mu_{\tilde{A}}(x) = 1 - \mu_A(x), \forall x \in U$
Intersection

- The fuzzy intersection, $A \cap B$, of two fuzzy sets $A$ and $B$, is given by $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$

### Diagram

- Young and middle-aged

Union

- The fuzzy union, $A \cup B$, of two fuzzy sets $A$ and $B$, is given by $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

### Diagram

- Young or middle-aged
Methodology

• Comprises a set of rules of the form
  – IF \( x \) is \( A \) AND \( y \) is \( B \) THEN \( z \) is \( C \)
    IF crisp_input matches fuzzy_input_term AND/OR ...
    THEN add fuzzy_output_term to fuzzy_output

• For each rule
  – for each antecedent
    • evaluate m.f. \( (\mu) \) of the crisp input value at the fuzzy term
  – combine the \( \mu \)'s using appropriate fuzzy operator
  – fire the consequence at strength of the resultant truth
    • add the output term to a (fuzzy) output set

• Interpret the output set in some way

Outline

1. Fuzzify inputs
2. Combine inputs
3. Perform implication
4. Aggregate output
5. (Defuzzify)
The Problem

- The result of Mamdani inference is one or more arbitrary output fuzzy set(s)
  - what is the meaning of such sets?
Defuzzification

• There are two principal forms of defuzzification
  – numeric defuzzification
  – linguistic defuzzification

• Numeric defuzzification
  – often, a single (crisp) number is required as output
    • e.g. fuzzy control
  – there are many different options
    • COG (centroid), mean-of-maxima, centre-of-area

• Linguistic defuzzification
  – a linguistic term representing the output set is found
  – some form of similarity or distance metric used

Centre of Gravity

• The imaginary balance point of the shape

\[ X_g = \frac{\sum_{i=1}^{N} (\mu_i \cdot x_i)}{\sum_{i=1}^{N} \mu_i} \]
Mean of Maxima

- The mean of the maximal membership grades

\[ x_m = \max_i (\mu_i) \]

Problems

- Information is lost
  - this is inevitable when reducing to a single number
  - e.g. these quite different shapes have same \( x_g \)
Linguistic Approximation

• A similarity measure is used to compute the distance between
  – the actual output set
  – an arbitrary collection of primitive terms, connectives and hedges
• A search is initiated to find the best term while limiting the complexity of the combination in order to produce comprehensible output
  – e.g. medium or high may be preferred to not extremely low or fairly medium or medium or fairly high
• Special level sets may also be included in search

Similarity Measures

• Euclidean distance
  \[ \delta^2 = \sum_{i=1}^{N} (\mu_i - \eta_i)^2 \]
  – where \( \eta_i \) is membership grade of linguistic term
  – minimum will determine the best match
• Degree of overlap
  \[ \gamma = \frac{A \cap B}{A \cup B} \]
  – maximum will determine the best match
Examples

- A is best approximated by low or high
- B is best approximated by mid
- C is best approximated by undefined (0/x)

However ...

A height of 6'3" has a membership of exactly 0.9: where is the fuzziness???
A height of 6’3” now has a membership of the type-2 fuzzy set tall of between 0.87 and 0.93.
Non-Stationary Fuzzy Sets

$\mu$

medium

height

Non-Stationary Fuzzy Sets

$\mu$

medium

height
The membership function wobbles from side to side (for example), so that each time we get a different $\mu$. 