Measuring the Directional Distance between Fuzzy Sets

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The outline of this presentation is as follows:

- Fuzzy sets
- Distance Measures
- Directional Distance Measures
- Non-normal fuzzy sets
- Non-convex fuzzy sets
- Future Work
- Questions
Fuzzy Sets

Crisp sets and Fuzzy sets

- Each element of a crisp set has a boolean membership value: 0 (False) and 1 (True)
- Each element of a fuzzy set has a membership value which lies between 0 and 1
- Fuzzy sets allows us to indicate a degree of uncertainty that an element belongs to the set.
Crisp set of “young”
Fuzzy set of “young”
Distance Measures

- A distance measure indicates how far apart two fuzzy sets are in their universe of discourse.
- For example, two fuzzy sets representing how delicious the food is at two restaurants, $A$ and $B$.
Distance Measures

- The distance between fuzzy sets is calculated using vertical or alpha (horizontal) cuts.
- Vertical slices are usually used to indicate similarity
- whereas alpha-cuts are usually used to indicate distance
Distance Measures

Vertical Slices
\[ d(A, B) = \frac{1}{n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| \]

Alpha cuts
\[ d(A, B) = \frac{1}{m} \sum_{i=0}^{m} \max\{|A_{\alpha l} - B_{\alpha l}|, |A_{\alpha r} - B_{\alpha r}|\} \]

Vertical Slices
(A, A) 0.0 0.538 0.396
(A, B) 0.0 3.495 6.262
(A, C)
The distance between fuzzy sets is measured by taking alpha-cuts.

The distance at alpha ($\alpha$) is measured using the Hausdorff metric:

$$ h = \max\{|\bar{A}_l - \bar{B}_l|, |\bar{A}_r - \bar{B}_r|\} $$
Distance Measures

Alpha-cuts

- The results of each alpha-cut are combined and weighted to create a single value of distance.

\[ d(A, B) = \frac{\sum_{\alpha=1}^{m} y_{\alpha} h(A_{\alpha}, B_{\alpha})}{\sum_{\alpha=1}^{m} y_{\alpha}} \]

- \( d(A, B) = 1.124, \quad d(B, A) = 1.124 \)

- But how can you tell which restaurant was rated more delicious than the other?
Altering the Hausdorff metric

The original Hausdorff metric \( h \), used to measure the distance between intervals

\[
h(\bar{A}, \bar{B}) = \max\{|\bar{A}_l - \bar{B}_l|, |\bar{A}_r - \bar{B}_r|\}
\]  

(1)

An amended measure to account for distance

\[
h(\bar{A}, \bar{B}) = \begin{cases} 
\bar{B}_l - \bar{A}_l, & \text{if } |\bar{B}_l - \bar{A}_l| > |\bar{B}_r - \bar{A}_r| \\
\bar{B}_r - \bar{A}_r, & \text{otherwise}.
\end{cases}
\]

(2)

Using (2), \( d(A, B) = 8 \) and \( d(B, A) = -8 \).
Movie Lens

Movie Lens is a dataset of film ratings given in the range 1 (poor) to 5 (great).

\[ D(SMB, SW) = 3.333 \]
\[ D(SW, SMB) = -3.333 \]

However, fuzzy sets are not always normally distributed.
Changing the films into a non-normal distribution

The membership value $\mu_A(x)$ now indicates the percentage of people who gave the value $x$ for the film $A$. 

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Non-normal Fuzzy Sets

Measuring the distance between non-normal fuzzy sets

- How do you measure the distance between a set and an empty set?
- How do you measure the distance between two empty sets?
Non-normal Fuzzy Sets

Measuring the distance between non-normal fuzzy sets

\[ h(A_{\alpha}, \emptyset) = h(A_{\alpha_k}, B_{\alpha_k}) \text{ where } \alpha_k \text{ is the } \alpha\text{-level at} \]
\[ \max\{|h(A_{\alpha}, B_{\alpha})|\} \forall \alpha \ A_\alpha \neq \emptyset \land B_\alpha \neq \emptyset. \]

- \( \alpha = 0.1, \ h = 1.44 \)
- \( \alpha = 0.2, \ h = 2.08 \)
- \( \alpha = 0.3, \ h = 2.36 \)
- \( \alpha = 0.4, \ h = 2.36 \)
- \( \alpha = 0.5, \ h = 2.36 \)
- distance \( = 2.261 \)

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Non-convex Fuzzy Sets

Measuring the distance between non-convex fuzzy sets

- The ratings for the film “All Dogs Go to Heaven 2” result in a non-convex fuzzy set.
- When taking an alpha-cut of a non-convex fuzzy set we no longer get an interval.
Measuring the distance between non-convex fuzzy sets

- $ADGH_\alpha$ is split into two intervals, $ADGH_{\alpha_1}$ and $ADGH_{\alpha_2}$.
- Next, we measure $h(ADGH_{\alpha_1}, SW_\alpha)$ and $h(ADGH_{\alpha_2}, SW_\alpha)$.
- Finally, calculate the average of these measures.
Future Work

Distance as a fuzzy set

- The distance between two fuzzy sets is represented as a fuzzy set.
- The greater the entropy of the result the more uncertain we are of the distance.
- Each alpha-cut of one fuzzy set is compared with each alpha-cut of the other fuzzy set.
Future Work

Distance as a fuzzy set

- Changing the direction reflects the resulting fuzzy set

![Graph showing membership values vs. rating and distance.]
Summary

To summarise

- We have created a distance measure which accounts for the direction as well as the magnitude of distance.
- We have solved measuring the distance of normal and non-normal fuzzy sets, as well as convex and non-convex fuzzy sets.
- We have demonstrated the applicability of the measure using real data.
Thanks you for listening.
Any Questions?