On Relationships between Primary Membership Functions and Output Uncertainties in Interval Type-2 and Non-Stationary Fuzzy Sets

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Abstract—The aim of this study was to explore relationships between the shape of the primary membership functions and the uncertainties obtained in the output sets for both non-stationary and interval type-2 fuzzy systems. The study was carried out on a fuzzy system implementing the standard XOR problem, in which either Gaussian or Triangular membership functions were employed, using a range of input values and recording the size of the output intervals obtained. It can be observed that the shape of the surfaces of the output intervals are related to the primary membership function and that the surface is divided into four roughly symmetrical parts. Furthermore, it can be observed that there are complex differences between the surfaces produced by interval type-2 systems and various kinds of non-stationary systems. Detailed differences between the output surfaces of uniformly distributed non-stationary systems are examined and the implications are discussed.

I. INTRODUCTION

All humans, including experts, exhibit variation in their decision making. Variation may occur among the decisions of a panel of human experts (inter-expert variability), as well as in the decisions of an individual expert over time (intra-expert variability). Up to now it has been an implicit assumption that expert systems, including fuzzy expert systems (FESs), should not exhibit such variation. While type-2 sets, which first introduced by Zadeh [1], capture the concept of introducing uncertainty into membership functions by introducing a range of membership values associated with each value of the base variable, they do not capture the notion of variability — a type-2 fuzzy inference system will always produce the same output(s) (albeit a type-2 set with an implicit representation of uncertainty) for given the same input(s). The class of a type-2 fuzzy set is determined by the secondary membership function. That is, if the secondary membership function simply takes the value zero outside the lower and upper bounds and 1 inside the bounds, then interval type-2 sets are obtained. If (type-1) fuzzy sets are used for the secondary membership functions, then the general type-2 fuzzy sets are obtained. This paper is only dealing with interval type-2 fuzzy sets and the interested reader is particularly referred to [2] and [3] for a summary tutorial and for more details.

Garibaldi et al [4], [5], [6], [7], [8] have been investigating the incorporation of variability into decision making in the context of FESs in the medical domain. In this work, Garibaldi proposed the notion of ‘non-deterministic fuzzy reasoning’ in which variability is introduced into the membership functions of a fuzzy system through the use of random alterations to the parameters of the generating function(s). Later, Garibaldi and Musikasuwan [9], [10] extended and formalised this notion through the introduction ‘non-stationary fuzzy sets’. Further details about non-stationary fuzzy sets and systems are provided in Section II.

The research presented here is continued from [11]. The experiments were designed by constructing the interval type-2 and non-stationary fuzzy systems using Gaussian or Triangular MFs as the primary MFs in a system to predict the results of the standard XOR problem, over a wide range of input values (21 × 21 = 441 pairs). The lower bound, mean, upper bound, and interval of the output for each system were computed and recorded. The experimental design is presented in Section III. The methods used in this study and the results obtained from the experiments are given in Sections IV and V. Finally, Section VI provides a discussion of the issues raised.

II. NON-STATIONARY FUZZY SETS AND SYSTEMS

The concept of non-stationary fuzzy sets have been formalised from the notion of ‘non-deterministic fuzzy reasoning’ and non-stationary fuzzy sets have previously been introduced to allow the modelling of variation in the membership value associated with a given value of the base variable of a fuzzy set by the use of random alterations to the parameters of the generating function(s) [9], [10]. The class of a non-stationary fuzzy set is determined both by which kind of non-stationarity is used (variation in location, variation in slope or noise variation) and by the form of perturbation function used to deviate the primary membership function. In this study we have used Normally distributed, Uniformly distributed, and Sine-based perturbation functions applied to both variation in location and variation in slope. Fig. 1 shows the mechanisms of how inferencing might be carried out using such non-stationary FLS’s.

At each instantiation a non-stationary fuzzy system operates as a type-1 fuzzy system. However, each particular instantiation may vary from the previous one by a small amount (caused by applying the perturbation function to the parameter of the MFs), to produce slightly different output(s). Hence the output(s) are recorded and then the process will be
repeated some number of times. Once all repeated processes have been completed, the final outputs will be calculated. Again, there is a range of options for how to determine the final output from the repeated runs.

Definition 1: A non-stationary fuzzy set, denoted \( \hat{A} \), is characterised by a membership function, \( \mu_\hat{A}(x,t) \), where \( (x) \in X \) and \( \mu_\hat{A}(x,t) \in [0,1] \) and \( t \) is a free variable, \emph{time} — the time at which the fuzzy set is instantiated, and is formally defined as:

\[
\hat{A} = \int_{x \in X} \mu_\hat{A}(x,t)/x, \mu_\hat{A} \in [0,1]
\] (1)

Any membership function may be used. Three main alternative kinds of non-stationarity have been proposed:

- Variation in location — i.e. small alterations to the centre point of the primary membership function
- Variation in slope — i.e. small alterations to the width of the primary membership function
- Noise variation — i.e. making small alterations (vertically) in the membership value of the membership function

Finally, the perturbation function refers to the function of time that will generate small changes in the base membership function. In theory, this could be a true random function — i.e. the membership function parameter could be a true random variable: hence the terminology of \emph{non-stationary} fuzzy sets. In general, it would be appear that any function of time might be used as the perturbation function, where the only restriction is that the membership function remains in bounds. Given that any measurement of time is arbitrary and relative, the actual set of functions that might be useful in practice is more restrictive. Any units might be used for time, \( t \), but the most natural would be to express time in seconds (s), in the absence of any good reason not to. Again, given that any physical notion of time is relative, any arbitrary point in time might be chosen as zero. The interested reader is particularly referred to [9], [10] for more details on non-stationary fuzzy sets and systems.

III. EXPERIMENTS

The aim of this study was to investigate the relationship between Gaussian and Triangular primary membership functions used in both non-stationary fuzzy sets and interval type-2 fuzzy sets, and the uncertainties obtained in the outputs.

Fuzzy systems were constructed to predict the output of the XOR truth value where both input variables can take any value in the range of [0,1]. All fuzzy systems consist of two input variables which are Input1 and Input2, one output variable which is Output, and four rules. Each variable consist of 2 Gaussian or Triangular MFs which are Low and High. In our previous work only a restricted range of input values were examined; specifically, the pairs (0.25,0.25), (0.25,0.75), (0.75,0.25) and (0.75,0.75). In this paper, each input is varied over the range [0,1] in increments of 0.05, giving a total of 441 pairs of Input1 and Input2 as [(0,0),(0,0.05), ..., (1,0.95), (1,1)]. The following 4 rules were used for all FESs:

1. IF Input1 is Low AND Input2 is Low THEN Output is Low
2. IF Input1 is Low AND Input2 is High THEN Output is High
3. IF Input1 is High AND Input2 is Low THEN Output is High
4. IF Input1 is High AND Input2 is High THEN Output is Low

There are three kinds of perturbation function that were used in this study, as follows:

- Sinusoidal function (where \( \omega = 127 \))
- Uniformly distributed function
- Normally distributed random function

The sinusoidal and uniformly distributed functions return numbers in the range \([-1,1]\), while the third (the Matlab \textit{randn} function) returns real numbers sampled from a Normal distribution with mean zero and standard deviation one.

A. Gaussian membership functions

The primary Gaussian MFs as shown in Fig. 2 were used and two kinds of variation were investigated, i.e. centre variation and width variation.

1) Non-stationary Fuzzy Systems: The non-stationary fuzzy sets were generated by replacing centre (\( c \)) or width (\( \sigma \)) with \( c = c + 0.05f(t) \) or \( \sigma = \sigma + 0.05f(t) \), where \( f(t) \) represents chosen perturbation function. The three different perturbation functions described above were used to generate the MFs. All terms (two inputs and one output) have two Gaussian membership functions, corresponding to meanings of Low and High. Low membership functions all have centre 0.1, High membership functions all have centre 0.9. Finally, the initial widths for all MFs for all terms were 0.5.

2) Interval type-2 Fuzzy Systems: Two interval type-2 FESs have also been designed, where the membership functions all have the same centre and width parameters as described above. The footprints of uncertainty of the type-2
MFs were created by deviating the parameters of the original type-1 MFs by a percentage of the universe of discourse of the variables that they are associated with. In the case of centre variation, the centre of lower and upper bounds MFs were defined by shifting the initial centre point both left and right for 5% of universe of discourse, as follows:

- Centre of $MF = c \pm 0.05$

Similarly, in the case of width variation, the width of lower and upper bounds MFs were defined by shifting the initial width both left and right for 5% as follows:

- Width of $MF = \sigma \pm 0.05$

B. Triangular membership functions

In this study four kinds of variation were investigated, i.e. centre variation, begin-point variation, end-point variation, and both begin and end points variation.

1) Non-stationary Fuzzy Systems: The Triangular primary MFs used throughout this case study to represent membership function are shown in Fig. 3. The non-stationary fuzzy sets were generated by replacing begin-point $a$ and/or end-point $b$, or centre-point $c$ in Fig. 3 with $a = a + 0.05 f(t)$, $b = b + 0.05 f(t)$, and $c = c + 0.05 f(t)$, where $f(t)$ represents the chosen perturbation function). Once again, the same three perturbation functions were used. Again, all terms (two inputs and one output) have two triangular membership functions, corresponding to meanings of Low and High. Low membership functions all have ordinary centre ($c$) 0.3, $a$ is 0.1, and $b$ is 0.5; High membership functions all have ordinary centre 0.7, $a$ is 0.5, and $b$ is 0.9.

2) Interval type-2 Fuzzy Systems: Similarly, four interval type-2 FESs were designed, where the membership functions all have the same parameters as described above. In the type-2 FES, the footprint of uncertainty of the MFs are created by deviating the parameters of the original type-1 MFs by a percentage of the universe of discourse of the variables that they are associated with. The four methods used to create these type-2 MFs were: by (i) varying the centre point around the original type-1 MF both left and right for 5% of the universe of discourse of the variable, as follows:

- Centre of lower and upper $MF = c \pm 0.05$

(ii) varying the begin-point ($a$) both left and right for 5% of the universe of discourse, as follows:

- Begin-point of lower and upper $MF = a \pm 0.05$

(iii) varying the end-point ($b$) both left and right for 5% of the universe of discourse, as follows:

- End-point of lower and upper $MF = b \pm 0.05$

and (iv) varying both begin and end points ($a$ and $b$) left and right for 2.5% of the universe of discourse, as follows:

- Begin-point of lower and upper $MF = a \pm 0.025$
- End-point of lower and upper $MF = b \pm 0.025$

IV. Methods

After all systems had been constructed, they were used to obtain the output of each pair of data sets (in total 441 pairs). The lower bound, mean, upper bound, and interval of the outputs were computed and recorded. In the case of interval type-2 systems, the lower and upper output bounds are those obtained directly from the systems; the mean was simply derived from the average of lower and upper outputs.

In the case of non-stationary systems, this process was repeated a fixed 30 times. For the Uniform and Sinusoidal perturbation functions, the lower bound, upper bound and the mean values were simply derived from interval obtained value, the maximum observed value and the mean of the observed values obtained in the 30 repeats, respectively. For the non-stationary systems utilising Normally distributed perturbation functions (only), the lower and upper bounds are derived from $m \pm s$, where $m$ is the mean of the outputs over the 30 repeats and $s$ is the standard deviation. Finally, the lower and upper bounds of the outputs from 441 input data pairs ([(0.0), (0.0,0.05), ..., (1.0,95), (1.1)]) were used to calculate the length of the interval of the outputs.

V. Results

Figs. 4 and Figs. 5 are surface plots showing the size of the interval obtained for the output, as a function of the two inputs, of the various systems utilising Gaussian membership functions. Fig. 4 (a) shows the size of the output interval obtained for the interval type-2 system with centre-variation, (b) shows that obtained for the non-stationary system utilising Normally distributed perturbation functions, (c) the non-stationary system using sinusoidal perturbation functions, and (d) using uniformly distributed perturbation functions. Fig. 5 shows the similar surfaces obtained for systems having width-variation.

Figs. 6 – 9 show similar plots obtained for systems featuring triangular primary membership functions exhibiting centre variation, begin-point variation, end-point variation and begin-end-point variation, respectively. Again, in each case the surfaces for (a) type-2, (b) Normally perturbed non-stationary, (c) sinusoidally perturbed non-stationary and (d) uniformly perturbed non-stationary systems are shown.

It can be seen that, in general, the shape of the surface obtained is similar for each of the different types of fuzzy system in each case (i.e. the shapes in Figs. 4 (a) – (d) are similar), although the magnitude varies. In order to explore this further, the difference between the intervals of the outputs of the interval type-2 fuzzy systems and the non-stationary fuzzy system with uniformly distributed perturbation functions were plotted. Fig. 10 shows the differences for the case of (a) Gaussian primary membership functions with centre variation, (b) Gaussian primary membership functions...
VI. DISCUSSION

The term ‘footprint of uncertainty’ (FOU) was introduced by Mendel to provide ‘a very convenient verbal description of the entire domain of support for all the secondary grades of a type-2 membership function’ [2]. We introduce a similar term, the ‘footprint of variation’ (FOV), as a similar verbal description of the area covering the range from the minimum to the maximum fuzzy sets which comprise the non-stationary fuzzy sets as shown in Fig. 12. For non-stationary fuzzy sets which are generated by Uniformly distributed and Sinusoidal perturbation functions (producing random values within \([-1, 1]\)), the maximum area of FOV will be equivalent to the FOU of interval type-2 fuzzy sets with the same amount of variation. Normally distributed perturbation functions generate random values within \([-\infty, \infty]\), and so an FOV defined as the union of all primary memberships would fill the entire universe of discourse. This kind of FOV will need further investigation.

In Fig. 4 and Fig. 5 (both for systems with Gaussian primary membership functions) it can be observed that the surfaces are (very approximately) comprised of four superimposed Gaussians. In the case of Fig. 4 the Gaussian-like shapes are located centrally on the x and y axes, while in Fig. 5 they are located on the corners. It would appear that there is some complex relationship between the FOV in non-stationary or FOU in type-2 systems and the size...
Fig. 5. The output’s intervals for width variation with Gaussian primary membership function in type-2 and non-stationary (NS) fuzzy systems.

(a) interval type-2 system  
(b) NS system with Normally distributed function

(c) NS system with Sinusoidal function  
(d) NS system with Uniformly distributed function

Fig. 6. The output’s intervals for variation in centre point with Triangular primary membership function in type-2 and non-stationary (NS) fuzzy systems.

(a) interval type-2 system  
(b) NS system with Normally distributed function

(c) NS system with Sinusoidal function  
(d) NS system with Uniformly distributed function
Fig. 7. The output’s intervals for variation in begin point with Triangular primary membership function in type-2 and non-stationary (NS) fuzzy systems

Fig. 8. The output’s intervals for variation in end point with Triangular primary membership function in type-2 and non-stationary (NS) fuzzy systems
Fig. 9. The output’s intervals for variation in both begin-end point with Triangular primary membership function in type-2 and non-stationary (NS) fuzzy systems

(a) interval type-2 system  
(b) NS system with Normally distributed function  
(c) NS system with Sinusoidal function  
(d) NS system with Uniformly distributed function

Fig. 10. The difference between the output’s intervals of interval type-2 fuzzy system and non-stationary fuzzy system with Uniformly distributed function

(a) Centre variation with Gaussian primary MF  
(b) Width variation with Gaussian primary MF  
(c) Centre variation with Triangular primary MF  
(d) Begin-end point variation with Triangular primary MF
of the interval in the output, although we cannot yet fully explain this relationship. Again, it can be observed that the surfaces are divided into four equivalent (symmetrical) parts. This might be expected because of the symmetrical nature of the four rule XOR problem and this observation will be further explored. Similarly, Figs. 6, 7, 8, 9 (triangular primary membership functions with centre variation) exhibits a similar four way symmetry, with vaguely triangular shapes almost appearing as projections of the primary membership functions. Fig. 9 also appears to exhibit four way symmetry, but have a more complex form.

An interesting observation from Figs. 7 and 8 is that Fig. 7 (a) and Fig. 8 (a) are not reflections of each other, as might be expected from the fact that varying the begin-point of the membership functions of all the variables is a reflection of varying the end-point of the membership functions. On examination, we found that Fig. 8 (a) can be obtained by rotating Fig. 7 (a) through 90° and then rotating each of the four quadrants through 90° separately. We believe that this is due to the lack of reflective symmetry in the rule set. That is, to obtain total reflection rule 1 would need to be changed to ‘IF Input1 is Low AND Input2 is Low THEN Output is High’, and so on.

There is an interested finding for the systems with triangular primary membership functions with one-side slope variation (begin point and end point) as shown in Figs. 7 and 8. It can be observed that there are plateau in the surface of the intervals of the outputs for the interval type-2 systems only, but this observation does not occur in the equivalent systems with centre and width (begin-end point) variation. At this stage, we cannot clearly explain the results obtained here, and more investigation and exploration is needed.

It can be seen from Fig. 10 that the difference of output’s interval between type-2 system and non-stationary systems with uniformly distributed perturbation functions is extremely small. Without performing a detailed mathematical analysis of the relationship between type-2 and non-stationary systems, it might be expected that uniformly distributed non-stationary systems with the same FOV as FOU of interval type-2 systems will match closely. It should be remembered that there is a stochastic element to the non-stationary systems, such that after 30 repeats an exact match would not be expected. As mentioned above, for the systems with one-side slope variation, Figs. 11 (a) and (b), there is a big difference between the output’s interval of type-2 system and non-stationary system with uniformly distributed perturbations. Once again, we are unsure of the significance of this finding at present. Non-stationary fuzzy systems may provide a mechanism for implementing a form of fuzzy reasoning which approximates general type-2 fuzzy inference in a simple, fast and computationally efficient manner. We are continuing investigations into the relationship between the two frameworks (non-stationary and type-2 systems).

ACKNOWLEDGMENT

This work is supported by the Royal Thai Government.

REFERENCES