

Automated Group Decision Support Systems Under Uncertainty: Trends and Future Research

Robert I. John¹, Shang-Ming Zhou¹, Jonathan M. Garibaldi² and Francisco Chiclana¹

¹Centre for Computational Intelligence, School of Computing,
De Montfort University, Leicester, LE1 9BH, UK
rij@dmu.ac.uk; smzhou@ieee.org; chiclana@dmu.ac.uk

²School of Computer Science,
University of Nottingham, Nottingham, NG8 1BB, UK
jmg@cs.nott.ac.uk

Abstract:

In the real world, group decision making is one of the most significant and omnipresent human decision making activities. The central problem of group decision making is to develop “fair” methods for aggregating individual alternatives (options, variants, etc.) to yield a consensus decision that is most acceptable to the group as a whole. It has become a subject of intensive research due to its practical and academic significance. But group decision making is innately uncertain, so a promising framework of decision making in groups is to consider uncertainty measures in the processes of model construction, so as to achieve optimal solutions in terms of best degrees of consensus of final decisions. This paper aims to provide an in-depth overview of group decision making incorporating uncertainty from the perspective of fuzzy sets (including type-1 and type-2 fuzzy sets). Some potential new research directions on fuzzy group decision making are suggested.

Keywords: Group decision making, uncertainty, fuzzy sets, type-2 fuzzy sets, aggregation, consensus.

I. Introduction

Group decision making is one of the most significant and omnipresent human decision making activities in real world applications. However, there are severe limitations in classical research on group decision making as indicated by the well-known *Arrow's Impossibility Theorem* [1]. Arrow proposes a qualitative setting characterised by a group of axioms such as non-dictatorship, unrestricted domain or universality, independence of irrelevant alternatives, and Pareto efficiency [1] [46], which are assumed to be reasonable requirements and should be satisfied by any aggregation type in group decision making. But *Arrow's Impossibility Theorem* states that there is no “rational” social choice (welfare) decision that satisfies all these “natural” or plausible requirements, each choice

function has at least one serious drawback. This means that in group decision making, it is impossible for a group decision outcome to satisfy every member by aggregating individual preferences into a group preference in a completely rational way. As a matter of fact, in practice, business and social analysts have already pointed out a number of problems with decision making under the requirement of full agreement [87]. In some cases, a too-strict requirement of consensus may (1) take a long time to reach, which is intolerable for some urgent matters; (2) encourage a small self-interested minority group dominating effectively the process of decision making; (3) encourage a “group think”, in which none of the group members individually believe that the final decision is wise. Hence, if the consensus is classically defined as the full and unanimous agreement of all the experts regarding all the available alternatives (options, variants, . . .), given the range of opinions obtained from a panel of experts, the chances of reaching a consensus on a decision or a decision-related task is rather low or impossible [38][46].

On the other hand, since the processes of decision making in the real world normally proceed in an environment in which the goals, constraints, and consequences of feasible options (alternatives, actions, . . .) are not precisely known, that is to say, inherent subjectivity, imprecision and vagueness in the articulation of opinions exist in these processes [5], the application of fuzzy sets to decision making is a major research topic in fuzzy logic community [13][20][29][33] [58]. What's more, for group decision making, the significance of introducing fuzzy set concepts into the process of group decision making lies in the fact that fuzzification of preferences, relations, decision making rules, etc. may alleviate difficulties in designing a social welfare function of the *Arrow Impossibility Theorem* type to arrive at an optimal consensus decision [46]. Some research has indicated that the

problem of social choice issued by *Arrow Impossibility Theorem* disappears in a cardinal setting of fuzzy context with introduced preference intensities due to additional degrees of freedom provided by any aggregation model [19][23]. As a matter of fact, full agreement is not necessary in real world group decision making, a “soft” consensus achieved by a “soft” majority such as *most, almost all, much more than a half* may be closer to the real human perception in decision making. This sort of “soft” majority can be characterised by fuzzy logic based linguistic quantifiers [10][55][78][79]. Hence, there is a growing stream of research on group decision making devoted to attacking the challenging task of reaching a consensus decision based on fuzzy logic techniques [3][6][9][26][53][54][57]. The purpose of this paper is to make an effort to provide an in-depth survey and analysis of state-of-the-art research in fuzzy logic based group decision making.

Currently, two general schemes can be used to model group decision making: the first is based on an aggregation-and-ranking procedure [14][40], by which the “best” alternative is chosen after a ranking process. The second one is based on a consensus-reaching oriented solution [38][39]. It is noted that the consensus-reaching oriented scheme also extensively involves the aggregation and ranking techniques. Moreover, the “best” alternative obtained by the aggregation-and-ranking scheme can be regarded as a “soft” consensus decision and the degree of consensus can be further obtained [6]. Therefore, in this paper we only focus on the consensus-reaching oriented scheme for modelling group decision making.

The organisation of this paper is as follows. Section 2 addresses the schemes of modelling uncertainty, mainly in terms of fuzziness. Aggregation operations for soft decision making are reviewed in Section 3, and Section 4 presents a survey of the methodologies for modelling group decision making from the perspective of fuzzy logic that aim at reaching optimal consensus decisions. Some open problems and potential future research directions are identified in Section 5. Section 6 concludes this paper.

II. Formalisation of Uncertainty

In real-world applications, human decision making is innately uncertain due to inherent subjectivity, vagueness in the articulation of human opinions, imprecision of measurement etc. Currently, there are many frameworks for dealing with uncertainty. Among them, the schemes of linguistic uncertainty (fuzziness) [106][107] and random uncertainty (randomness) [25][51] [70] are most widely used. The former describes the *event class ambiguity*, a type of deterministic uncertainty, so the fuzzy membership grade measures the degree to which an event occurs, not whether this event occurs. The random uncertainty describes an event following a non-deterministic pattern: whether or not an event occurs. Probability theory handles this question. When the two sorts

of uncertainties happen simultaneously in a system, fuzzy random sets, fuzzy random variables, fuzzy probability and non-deterministic fuzzy set are usually used to characterise them [2][11][81][91][75][76][77]. Fuzzy randomness simultaneously describes objective and subjective information as a fuzzy set of possible probabilistic models over some range of imprecision. This generalized uncertainty model contains fuzziness and randomness as special cases. However, this paper focuses on review and analysis of state-of-the-art research in the field of group decision making from the perspective of fuzziness.

A. Type-1 Fuzzy Sets

The traditional fuzzy set was proposed by Zadeh in 1965 [106], which is sometimes called a type-1 fuzzy set to help distinguish it from a type-2 fuzzy set proposed by Zadeh later in 1975 [107]. In the literature, unless otherwise stated, the fuzzy sets described are usually type-1 fuzzy sets.

In classical set theory, the membership of an element that belongs to a set is assessed according to binary conditions: either it belongs or does not belong to the set. So the boundary of the classical set is crisp. Fuzzy set theory allows the gradual assessment of the membership of an element belonging to a set with the aid of a membership function. Formally, a fuzzy set is defined as follows (see the Figure 1):

Definition 1 Let X represent the domain of discourse. A fuzzy set A on X is a set expressed by a characteristic function $\mu_A: X \rightarrow [0, 1]$ that measures the membership grade of the elements in X belonging to the set A , i.e.,

$$A = \{ (x, \mu_A(x)) \mid \forall x \in X, \mu_A(x) \in [0, 1] \} \quad (1)$$

where $\mu_A(x)$ is called membership function of the fuzzy set A , and may be continuous, or the set contains only discrete elements assessed by membership grades.

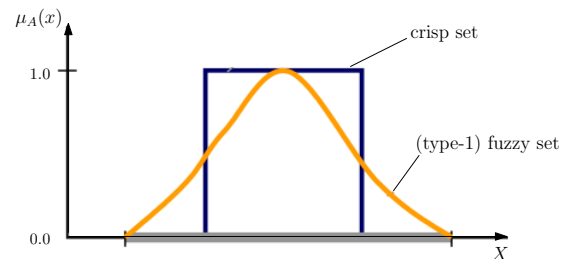


Figure 1: (Type-1) fuzzy set vs crisp set

Currently the type-1 fuzzy set is a widely prevalent paradigm used to characterise the linguistic uncertainties of fuzzy systems in terms of exact and crisp values in $[0, 1]$.

B. Type-2 Fuzzy Sets

Exact and crisp values in $[0, 1]$ are used to measure the degrees of membership of elements belonging to a type-1 fuzzy set. One fundamental question is: can we use certain values to measure an uncertain quantity when we think this quantity is uncertain? Zadeh [109] presents a powerful argument that perceptions (for example, perceptions of size, safety, health and comfort) cannot be modelled by traditional mathematical techniques which are certain and precise. However, a type-1 fuzzy set is precise in nature. Recognising the limitation of type-1 fuzzy sets in representing linguistic uncertainties by crisp values, in 1975 Zadeh introduced the type-2 fuzzy set whose grades of membership degree are type-1 fuzzy sets [107]. However, the concerns about type-1 fuzzy sets still apply to type-2 fuzzy sets as we still need to specify them exactly, which again seems counter-intuitive [48][63][65]. To overcome this, we need third order, fourth order descriptions, etc. Interestingly, Kreinovich and Nguyen [52] proved that the uncertain information contained in the first and second order fuzzy sets is sufficient to reconstruct all higher order fuzzy sets as well. In this sense, the type-2 fuzzy set description is sufficient in a computer representation of uncertainty. This result is somewhat similar to the well-known result that a Gaussian distribution can be uniquely determined by its moments of first and second orders, and all higher order moments can be uniquely reconstructed from the moments of the first two orders [52].

Mendel and John [64] have formally characterised a type-2 fuzzy set by a two-dimensional membership function.

Definition 2 A type-2 fuzzy set, denoted \tilde{A} , is characterised by a type-2 membership function $\mu_{\tilde{A}}(x, u)$, where $x \in X$ and $u \in J_x \subseteq [0, 1]$, i.e.,

$$\tilde{A} = \{ ((x, u), \mu_{\tilde{A}}(x, u)) \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1] \} \quad (2)$$

in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$, X is the universe of discourse. \tilde{A} can also be expressed as

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u) \quad J_x \subseteq [0, 1] \quad (3)$$

where \int denotes union over all admissible x and u .

Figure 2 shows an example of a type-2 membership function for x and u discrete with $X = \{1, 2, 3, 4, 5\}$ and $U = \{0, 0.2, 0.4, 0.6, 0.8\}$.

Recently, type-2 fuzzy sets and system modelling have been extensively investigated with many successful applications in various domains where uncertainties occur such as in decision making [12][95], diagnostic medicine [42], signal processing [49][62][112], traffic forecasting [59], mobile robot control [36], clustering analysis [41], and pattern recognition [67][110].

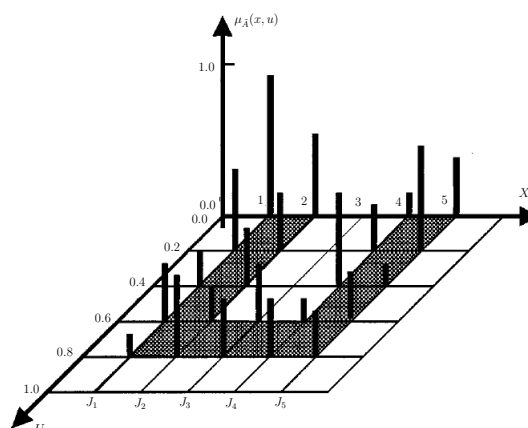


Figure. 2: Example of a type-2 fuzzy set[64]

III. Aggregation Operations for Soft Decision Making

Aggregation is a necessary process that has to be solved well in a group decision making problem. The aim of aggregation is to combine individual experts' preferences values into a group collective one in a proper way so that the final result of aggregation takes into account all the individual values. Currently, at least 90 different families of aggregation operators have been extensively studied [24][50][58][94]. In this section, we only focus on the state-of-the-art in several of them. It is known that experts' preferences values may be numerical values or non-numerical values like linguistic scale, such as small, large, average, good etc.. Hence, we review some techniques that attempt to fulfill the commitments of aggregating numerical values and linguistic scales separately.

A. Fuzzy Aggregation Operators for Numerical Values

Aggregation operators can be roughly divided into three classes: conjunctive operators, disjunctive operators, and averaging operators [35][96]. Each of the three classes possess very distinct behaviours. Conjunctive operators combine values so that the final result of aggregation is high if and only if all the individuals are high. The logical "and" operator is such a conjunctive aggregation operator. Generally, triangular norms are suitable for doing conjunctive aggregation. Disjunctive operators combine values as an "or" operator, by which the aggregation result is high if some (at least one) values are high. Triangular conorms are commonly used to perform disjunctive aggregation. Whereas averaging operators lie between conjunctive operators and disjunctive operators. One property of averaging operators is that they perform the aggregation in a compensative manner, i.e., low values can be compensated by high values so that the final aggregation result locates between minimum and maximum of all the individuals.

However, it is not an easy task to design an appropriate ag-

gregation operator. An aggregation operator should possess the following properties which are required for aggregation: idempotence, continuity, monotonicity, commutativity, compensativeness, and associativity. Details of these properties and mathematical descriptions can be found elsewhere [29][35].

1) γ -family of operators

Zimmermann and Zysno developed a family of compensatory operators for aggregating memberships by mixing a t-norm and a t-conorm to a proportion γ in the interests of producing a kind of compensation between criteria [114][115]. Zimmermann presents a list of rules about how to select the “best” operator in practice and addresses the importance of selecting an appropriate operator, given the membership functions that are to be aggregated. This family of operators is determined by a single parameter γ defined in the following way,

$$\mu_A(x) = \left(\prod_{i=1}^n \mu_{B_i}(x) \right)^{1-\gamma} \left(1 - \prod_{i=1}^n (1 - \mu_{B_i}(x)) \right)^\gamma \tag{4}$$

where A_i and B_i are aggregated objects in the form of fuzzy sets, $0 \leq \gamma \leq 1$. It can be seen that the aggregated membership grade $\mu_A(x)$ depends on the value of γ . Two operators-*product operator* and *algebraic sum operator* emerge when $\gamma = 0$ and $\gamma = 1$ respectively. If these two operators are to represent noncompensatory “and” and full compensatory “or” respectively, then the parameter γ can be interpreted as a measure of the grade of compensation [114].

In equation (4), it is assumed that the individual components B_i are equally important, which can be considered as a homogeneous aggregation in group decision making, i.e., in which all the experts are equally treated in the aggregating process. For instance, in practice, in case of no explicit importance degrees of individual preferences to be aggregated available, all the experts are usually said to be equally important. However, once an expert’s opinion provides information that discriminates him/her as not equally important, it may be more reasonable to give more importance to the experts who provide more consistent information in group decision making, which leads to a heterogeneous group decision making. Hence it is necessary to revise (4) for heterogeneous group decision making problems by assigning different weights for B_i as follows,

$$\mu_A(x) = \left(\prod_{i=1}^n (\mu_{B_i}(x))^{w_i} \right)^{1-\gamma} \left(1 - \prod_{i=1}^n (1 - \mu_{B_i}(x))^{w_i} \right)^\gamma \tag{5}$$

One of the advantages of the above Zimmermann’s γ -family of operators is its easy computation. However, if there exist some points x with $\mu_{B_i}(x) = 0$, which is a case often happening in practice, then $\mu_A(x) = 0$ no matter what γ is. In other words, Zimmermann’s γ -family of operators does not

give any compensation if any of the individuals to be aggregated are estimated to be zero [21].

2) The family of Yager’s OWA operators

In 1988, Yager introduced a new aggregation technique based on ordered weighted averaging (OWA) operators [97]. Since then, the OWA based aggregation strategies have been widely investigated and achieved many successful applications in many areas, such as decision making [9][18][97][98], fuzzy control [103][104], market analysis [105], image compression [69] etc..

Definition 3 An OWA operator of dimension n is a mapping $\phi : R^n \rightarrow R$, that has associated a set of weights in a vector $w = (w_1, \dots, w_n)^T$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. Given $a = [a_1, \dots, a_n] \in R^n$, ϕ is defined as follows,

$$\phi(a) = \phi(a_1, \dots, a_n) = \sum_{i=1}^n w_i a_{\sigma(i)} \tag{6}$$

where $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$, $\forall i = 1, \dots, n - 1$, i.e., $a_{\sigma(i)}$ is the i th largest element in the set $\{a_1, \dots, a_n\}$.

Generally speaking, the OWA operator based aggregation consists of three steps. The first step is to re-order the input arguments in descending order, in which the element a_i is not associated with a particular weight w_i , but rather w_i is associated with a particular ordered position of an aggregated object. This re-ordering step introduces a nonlinearity into the aggregation process; the second step is to determine the weights for the operator in a proper way; in the third step, the OWA weights are used to aggregate these re-ordered arguments.

OWA operators are idempotent, continuous, monotonic, commutative, and compensative. It is not difficult to prove that “min” and “max” operators, i.e., by setting $w_* = (0, 0, \dots, 0, 1)^T$ and $w^* = (1, 0, \dots, 0, 0)^T$ separately are the lower and the upper limit for any OWA operators. Like the product operator and algebraic sum in the Zimmermann’s γ -family of operators, the “min” and “max” in the family of Yager’s OWA operators may represent the connectives “and” and “or”. One property of compensative connectives is that a higher degree of satisfaction of one criterion can compensate for a lower degree of satisfaction of another criterion. The OWA operators can vary continuously from the “and” (min) to “or” (max) aggregation. Therefore, Yager introduces the *degree of orness* and the *degree of andness* associated with each weight vector as follows individually [97],

$$orness(w) = \frac{1}{n-1} \sum_{i=1}^n (n-i)w_i \tag{7}$$

$$andness(w) = 1 - orness(w) \tag{8}$$

The degree of orness characterises the degree to which the aggregation is like an *or* operation. $orness(w)$ indicates the degree of compensation, which is similar with the degree of compensation γ discussed in the previous subsection. The important difference is that many different OWA operators, i.e., different weight vectors w , may have the same degree of orness $orness(w)$, whilst each γ only corresponds to one γ -operator.

In OWA based aggregation, the choice of operator, i.e., the identification of the OWA weights is crucial, because the OWA weights reflect the decision makers' desired agenda for aggregating the criteria. A number of methods have been developed to fulfill this task. These methods can be roughly classified into three categories: the ones based on semantic considerations, the ones based on entropy measure as objective function, and the ones based on learning from observation data.

Identification of OWA operators based on semantic considerations

One popular method for identifying OWA weights is via linguistic quantifiers initially proposed by Yager [97][96]. Let $\{A_1, \dots, A_n\}$ be a set of criteria and x be an object. $A_i(x)$ represents the degree to which criterion A_i is satisfied by x . It can be seen that the aggregation $E_1(x) = \min(A_1(x), \dots, A_n(x))$ is to find out the degree to which x satisfies "all" the criteria, whilst the aggregation $E_2(x) = \max(A_1(x), \dots, A_n(x))$ to find out the degree to which x satisfies "at least one of" the criteria. However, in many applications these two extreme situations "all" or "at least one of" may be too strict. A "soft" majority or minority such as *most*, *almost all*, *at least half* may be closer to the real human perception in decision making. Interestingly, based on Zadeh's concept of linguistic quantifiers [108], some OWA operators can be obtained to characterise these "soft" majority or minority operations [97][96]. Specifically speaking, given a linguistic quantifier Q such as *most*, for the collection of the criteria $\{A_1, \dots, A_n\}$, one can consider an aggregation of these criteria based on an imperative such as *most of the criteria should be satisfied*. Yager called these as quantifier guided aggregations.

For a non-decreasing quantifier Q guided OWA operator $\phi_Q(a_1, \dots, a_n)$, the associated weights are determined in the following way,

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \quad i = 1, \dots, n \quad (9)$$

and the two conditions about weights: $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$ are satisfied. Figure 3 illustrates the method for determining the OWA associated weights via a regular non-decreasing quantifier. In [101], Yager and Filev indicated how the regular non-increasing quantifiers and unimodal quantifiers can be used to guide OWA aggregations.

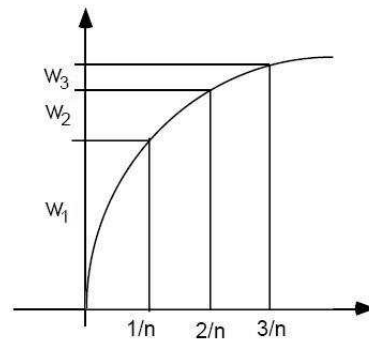


Figure. 3: Determining OWA weights via a quantifier

Identification of OWA operators based on entropy measure as objective functions

In order to evaluate how much of the information in the arguments is used during aggregation by an OWA operator, Yager introduced the dispersion or entropy measure associated with a weighting vector [97]

$$disp(w) = - \sum_{i=1}^n w_i \ln w_i \quad (10)$$

The larger the $disp(w)$ is, the more information is provided by the arguments during aggregation. An important issue is to determine a special class of OWA operators that make use of the maximum information available during aggregation for a given degree of compensation. This approach suggested by O'Hagan is based on the solution of the following mathematical programming problem [73]:

$$\max_w - \sum_{i=1}^n w_i \ln w_i \quad (11)$$

subject to

$$\begin{cases} orness(w) = \beta \\ \sum_{i=1}^n w_i = 1 \\ 0 \leq w_i \leq 1 \quad (i = 1, \dots, n) \end{cases} \quad (12)$$

Using the method of Lagrange multipliers, an analytic form of optimal weights for the above constraint optimisation problem can be obtained [32][34].

Identification of OWA operators based on learning from data

Empirical fit is one of the criteria that Zimmermann presented for selecting an appropriate aggregation operator [114]. Given the empirical data with K samples $\{a^{(k)}, d^{(k)}\}_{k=1}^K$, in which $a^{(k)} = (a_1^{(k)}, \dots, a_n^{(k)})^T$ is called the arguments, $d^{(k)}$ is an associated single value called the aggregated empirical value. The goal is to determine an

OWA operator that fits well these observations on the decision makers' performance by learning the weights. In this way, the choice of an OWA operator problem is usually transformed into a sort of regression problem, in which the optimal weights for OWA aggregation are obtained by minimizing an error function. We formulate the optimization problem as follows,

$$\min_w \left(\sum_{k=1}^K \phi \left(a_1^{(k)}, \dots, a_n^{(k)} \right) - d^{(k)} \right)^2 \quad (13)$$

subject to

$$\begin{cases} \sum_{i=1}^n w_i = 1 \\ 0 \leq w_i \leq 1 \quad (i = 1, \dots, n) \end{cases} \quad (14)$$

So the choice of an OWA operator becomes a quadratic optimization problem. However, due to the constraints involved, the solution of the above quadratic problem is not as simple as the one of traditional linear regression problem. To circumvent the constraints on w , Filev and Yager first transformed the domain of w to an unrestricted domain, then a gradient descent algorithm was used to minimize the transformed error function [30][31][100]. In [4], Beliakov described a method of directly solving the restricted linear least squares problem via a linear nonnegative least squares technique. Torra presented an alternative approach based on active set methods in [88] and applied OWA operators to data modelling and information extraction via active set methods [89].

B. Linguistic Information Aggregation

The aggregation methods reviewed in the above two subsections are centred on manipulation of numerical values from measurements, in which the aggregation results are crisp values. However, in real world applications, humans exhibit remarkable capability to manipulate perceptions, such as perceptions of distance, size, weight, likelihood, and other characteristics of physical and mental objects, without any measurements and any computations [109]. Specifically, in group decision making, based on the linguistic judgments for each expert on each alternative with respect to each criterion, a promising framework of evaluating the overall judgments of the alternatives for each expert is via an aggregation of the linguistic judgments on each alternative with respect to the criteria [9]. Currently there are two main paradigms for aggregating linguistic information.

The first paradigm for aggregating linguistic information is to directly work on linguistic labels, the only requirement is that these linguistic labels should satisfy an ordinal relation. In [9], Bordogna *et al* proposed a method of modelling consensus linguistically in group decision making, in which both the experts' evaluations of alternatives and the degree of consensus are expressed linguistically and overall

linguistic performance evaluation is completed through a linguistic OWA operator based aggregation. Another linguistic OWA operator proposed by Herrera *et al* [37] integrates the OWA operator [97] and a convex combination method of linguistic labels. Moreover Ben-Arieh and Chen presented an improved linguistic aggregation operator in [6]. One advantage of such a scheme of directly aggregating linguistic labels without considering the expression of the linguistic terms lies in its high computing efficiency due to its symbolic aggregation in nature. However, the precision of the linguistic operations is an issue: in some cases, this scheme may yield a solution set with multiple alternatives for decision makers to choose, rather than a single one. Another matter is that most of the existing methods based on this scheme use the traditional OWA operator which aggregates crisp numbers.

The second approach is to aggregate linguistic information via operations performed on their associated membership functions in fuzzy arithmetic and a linguistic approximation process based on Zadeh's Extension Principle. In [8], arithmetic operations on linguistic terms in the representations of trapezoidal fuzzy numbers were obtained by using the Extension Principle, and triangular norms were used to aggregate the fuzzy numbers. In [66], in order to evaluate an overall linguistic value of every expert's performance for each alternative with respect to the criteria, a weighted average of the membership function values associated with the linguistic performance labels was first computed, then the aggregation result was translated into linguistic terms via linguistic approximation. In the case of decision making applications, the importance weights in aggregation may be uncertain rather than represented by precise numerical values. A new type of weighted averaging operator that can yield a linguistic evaluation on overall performance of each expert through aggregation has been proposed [22], i.e., fuzzy weighted average defined as follows,

$$\phi(A_1, \dots, A_n) = \frac{W_1 A_1 + \dots + W_n A_n}{W_1 + \dots + W_n} \quad (15)$$

in which A_i are fuzzy numbers, and W_i are fuzzy sets representing the importance weights. So, according to the Extension Principle, the final aggregation result is a fuzzy set, which is quite different from the classic aggregation operators. Recently, Wu and Mendel [93] proposed the type-2 fuzzy weighted average, in which interval type-2 fuzzy sets instead of type-1 fuzzy sets were used to model the linguistic weights and aggregated objects.

However, the fuzzy weighted averaging operator implies preferential independence of the experts' points of view. Preference is an important issue in soft decision making, and has found significant interest in various fields such as economic decision making, social choice theory, operations research, databases, and human-computer interaction. One way of avoiding this independence condition is to generalise the traditional OWA operator as an aggregation operator for type-1 fuzzy sets. Zhou *et al* [113] proposed a new kind

of OWA operator, the type-1 OWA operator, that is able to aggregate the linguistic information modelled by fuzzy sets with linguistic weights in the form of fuzzy sets as well. Because Yager’s OWA operator is nonlinear as opposite to the weighted averaging operator, a linear operator, so the type-1 OWA operator is significantly different from the fuzzy weighted average operator. Let $F(X)$ be the set of type-1 fuzzy sets defined on the domain of discourse X .

Definition 4 Given n linguistic weights $\{W_i\}_{i=1}^n$ in the form of type-1 fuzzy sets defined on the domain of discourse $[0, 1]$, an associated type-1 OWA operator of dimension n is a mapping Φ ,

$$\Phi: F(X) \times \dots \times F(X) \longrightarrow F(X)$$

$$(A_1, \dots, A_n) \mapsto G$$

that aggregates type-1 fuzzy sets $\{A_i\}_{i=1}^n$ in the following way,

$$\mu_G(y) = \sup_{\substack{\sum_{k=1}^n \bar{w}_i a_{\sigma(i)} = y \\ w_i \in U, a_i \in X}} \left(\mu_{W_1}(w_1) * \dots * \mu_{W_n}(w_n) \right) * \mu_{A_1}(a_1) * \dots * \mu_{A_n}(a_n) \tag{16}$$

where $*$ is a t -norm operator, $\bar{w}_i = \frac{w_i}{\sum_{i=1}^n w_i}$, and $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ a permutation function such that $a_{\sigma(i)}$ is the i -th largest element in the set $\{a_1, \dots, a_n\}$.

From the above definition, it can be seen that $\Phi(A_1, \dots, A_n) = G \in F(X)$ is a type-1 fuzzy set defined on X . Moreover, Zhou *et al* proposed a method of inducing linguistic weights for type-1 OWA operators based on a new kind of linguistic quantifier, the type-2 quantifiers that are modelled by type-2 fuzzy sets. Also a procedure for performing type-1 OWA operations in dealing with the problem of *over-partition* of the input space was presented [113]. Figure 5 shows an example of aggregating four type-1 fuzzy sets by type-1 OWA operator using the linguistic weights in Figure 4, for instance, $\mu_G(2.0) = 0.85$.

In summary, inspired by the human crucial capability to perform various physical and mental tasks without any measurements, development of aggregation operators that can directly manipulate the linguistic terms is particularly suitable for applications within the scope of Zadeh’s approach to computing with words [99][109].

IV. Fuzzy Group Decision Making Modelling Methodologies

In this section, we review the methodologies for fuzzy group decision making. First let $X = \{x_1, \dots, x_n\}$ be a finite set of alternatives, $E = \{e_1, \dots, e_m\}$ represent a group of

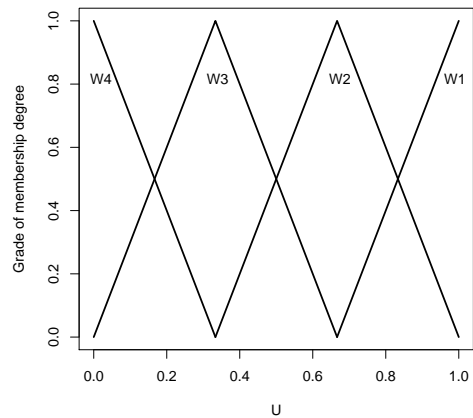


Figure. 4: Linguistic weights using in type-1 OWA (from right to left): W_1, W_2, W_3 and W_4

experts, and $S = \{s_1, \dots, s_g\}$ be linguistic term set with the ordered structure such that $s_i < s_j \Leftrightarrow i < j$.

A. Five Preference Representations

Although in decision making, experts have many ways of representing their opinions on the alternatives, a common way is based on their preferences over the set of the alternatives. Currently, there are five ways available to represent the experts’ preferences in literature.

1) Preference Ordering of Alternatives

In this way, each expert e_k represents his/her preferences on $\{x_1, \dots, x_n\}$ from best to worst in an ordered vector $O^k = \{o^k(1), \dots, o^k(n)\}$, where $o^k(\cdot)$ is a permutation function on the index set $\{1, \dots, n\}$ [15][83].

2) Fuzzy Preference Relation

In this way, each expert expresses his/her preferences on X by a fuzzy relation $P^k \subset X \times X$ [7][43][74], whose membership function $\mu_{PK}: X \times X \rightarrow [0, 1]$ is defined as

$$\mu_{PK}(x_i, x_j) = \begin{cases} 1 & \text{if } x_i \text{ is definitely preferred to } x_j \\ c \in (0.5, 1) & \text{if } x_i \text{ is slightly preferred to } x_j \\ 0.5 & \text{in case of indifference} \\ c \in (0, 0.5) & \text{if } x_j \text{ is slightly preferred to } x_i \\ 0 & \text{if } x_j \text{ is definitely preferred to } x_i \end{cases} \tag{17}$$

This fuzzy preference relation is usually assumed to be reciprocal in that $p_{ij}^k + p_{ji}^k = 1$ and $p_{ii}^k = 0.5$ [74][86], where $p_{ij}^k = \mu_{PK}(x_i, x_j)$.

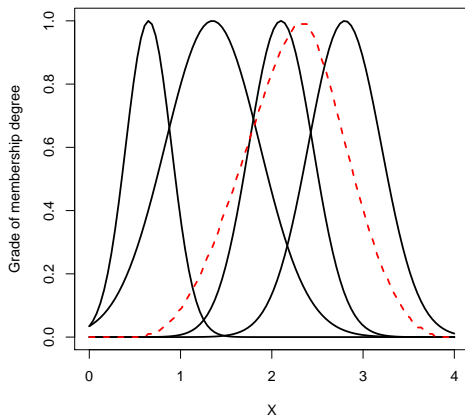


Figure. 5: Aggregation of type-1 OWA operator with linguistic weights given in Figure 4-dashed line:aggregation result; solid lines: to be aggregated fuzzy sets

3) *Multiplicative Preference Relation*

In this way, the expert’s preferences on alternatives X are expressed [82] by a preference fuzzy relation $U^k = (u_{ij}^k) \subset X \times X$, where u_{ij}^k is a ratio of the preference intensity of x_i to that of x_j . u_{ij}^k can be interpreted as x_i is u_{ij}^k times as good as x_j . Saaty suggested to represent u_{ij}^k in the scale of 1 to 9 [82], i.e., $u_{ij}^k = 1$ indicates indifference between x_i and x_j , $u_{ij}^k = 9$ indicates x_i is definitely preferred to x_j , and $u_{ij}^k \in \{2, \dots, 8\}$ denotes the intermediate evaluation. This preference relation is usually assumed to be multiplicatively reciprocal in that $u_{ij}^k \cdot u_{ji}^k = 1$.

4) *Utility Function*

In this way, each expert uses utility values $J^k = \{J_i^k\}_{i=1}^n$ to express his/her preferences on X [86], where $J_i^k \in [0, 1]$ denotes the utility evaluation to the alternatives x_i according to the expert e_k ’s preference.

5) *Linguistic Preference Values*

In this way, each expert uses linguistic values $L^k = \{l_i^k | l_i^k \in S\}_{i=1}^n$ to express his/her assessments on X [40][66], where l_i^k denotes the linguistic assessment on the alternative x_i by the expert e_k .

Among the above five preference representations, fuzzy preference relation p_{ij}^k is the most widely used one in practical applications[7][38][43][46][74][86]. So some researchers developed methods to transform other ways of preference representations into the fuzzy preference relation [15][16].

B. *Methodologies of Group Decision Making under Fuzziness*

General speaking, in the consensus-reaching oriented scheme, two processes are carried out before arriving at the final decision [29][38] [46]: the *consensus process* and the *selection process*. A consensus process can be considered as a dynamic iterative group discussion process that is coordinated by a moderator helping the experts to bring their opinions closer. Since full agreement is counterintuitive in real world group decision making, a “soft” consensus may be closer to the real human perception in decision making. What the moderator tries to do is to reach a high degree of consensus and make sure in coming to the consensual decision, everyone’s opinion on the matter is valued. The selection process is used to yield the solution set of alternatives from the opinions on the alternatives provided by aggregation and/or ranking. Clearly, it is expected that a high degree of consensus could be achieved in the group of experts with fuzzy representations. We now review some techniques that could potentially be utilised in the above processes to attack the group decision making problems under the environments of uncertainty.

Kacprzyk, Fedrizzi and Nurmi [43][47] proposed a methodology of group decision making under fuzziness and fuzzy majority which takes in fuzzy preference relations over alternatives and yields an alternative output (or a set of alternatives) that are acceptable to most individuals. This methodology has been extensively investigated by themselves and other researchers [28][43][47][46][61][80]. In this methodology, each individual e^k is required to provide his/her pairwise comparison between the alternatives for constructing the fuzzy preference matrix P^k , a degree of support to each alternative from most decision makers is obtained by aggregation before final decision is made and a degree of consensus is computed. Two lines of reasoning are available in this methodology to solve the pairwise group decision making problem: a direct approach and an indirect approach [43][47]. Marimin *et al* extended this methodology including the direct approach and the indirect approach by allowing the decision makers to express their preference relations in linguistic labels rather than in numerical values [61]. In this approach, preference relations are represented internally using triangular fuzzy numbers and unordered preference relations are allowed to be aggregated by using the neat OWA operators [96]. In [80], Prodanovic and Simonovic combined the direct approach in the Kacprzyk-Fedrizzi-Nurmi methodology with a fuzzy compromise programming to solve the problem of multiple criteria group decision making. In the following, we briefly review the direct approach to reasoning for solving the pairwise group decision making by this scheme.

In the direct approach, based on the so-called fuzzy cores, a solution is derived (without any intermediate steps) just from the set of individual fuzzy preference relations. First, for each individual e^k , h_{ij}^k is calculated to see whether the

alternative x_i defeats (in pairwise comparison) alternative x_j ($h_{ij}^k = 1$) or not ($h_{ij}^k = 0$), i.e.,

$$h_{ij}^k = \begin{cases} 1 & \text{if } p_{ij}^k < 0.5 \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

then, the degree of support to alternative x_j by individual e^k is calculated as follows,

$$h_j^k = \frac{1}{n-1} \sum_{i=1, i \neq j}^n h_{ij}^k \quad (19)$$

which is clearly the extent, from 0 to 1, to which individual e^k is not against alternative x_j ($h_j^k = 0$: definitely not against, $h_j^k = 1$: definitely against). Then we calculate the following averaging value

$$h_j = \frac{1}{m} \sum_k^m h_j^k \quad (20)$$

which expresses to what extent, from 0 to 1 as in (19), all the individuals are not against alternative x_j . Then a linguistic majority concept is used to approximate the final solutions,

$$v_Q^j = \mu_Q(h_j) \quad (21)$$

which is to what extent, from 0 to 1 as before, Q (say, *most*) individuals are not against alternative x_j , where Q is a linguistic quantifier “*most*” defined in Figure 4 (a). The final result (fuzzy- Q -core) is expressed as

$$C_Q = v_Q^1/x_1 + \dots + v_Q^n/x_n \quad (22)$$

which is interpreted as a fuzzy set of alternatives that are not defeated by Q (say, *most*) individuals.

Similarly, fuzzy α/Q -core and fuzzy s/Q -core can also be obtained. The fuzzy α/Q -core is determined by updating (18) into

$$h_{ij}^k(\alpha) = \begin{cases} 1 & \text{if } p_{ij}^k < \alpha < 0.5 \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

other steps remain the same. $(1 - \alpha)$ represents a degree of defeat to which x_i defeats x_j . The fuzzy α/Q -core obtained in this way can be interpreted as a fuzzy set of alternatives that are not sufficiently (at least to a degree $1 - \alpha$) defeated by Q (say, *most*) individuals. For fuzzy s/Q -core, only the equation (18) is updated into

$$h_{ij}^k = \begin{cases} 2(0.5 - p_{ij}^k) & \text{if } p_{ij}^k < 0.5 \\ 0 & \text{otherwise} \end{cases} \quad (24)$$

by introducing the strength of defeat. The obtained fuzzy s/Q -core can be interpreted as a fuzzy set of alternatives that are not strongly defeated by Q (say, *most*) individuals.

In order to alleviate some “rigidness” of the conventional concept of consensus, i.e., full consensus occurs only when all the individual experts agree as to all the issues, a degree of consensus model was proposed by Kacprzyk [44],

then advanced by Fedrizzi and Kacprzyk [27], and Kacprzyk and Fedrizzi [45]. By introducing linguistic quantifiers into group decision making, in the new degree of consensus proposed in [47][46], full consensus may occur when most of the individual experts agree as to almost all (of the relevant) issues (alternatives, options).

Basically, the new degree of consensus is derived in three steps [47][46]:

- For each pair of individuals, a degree of agreement as to their preferences between all the pairs of alternatives is derived;
- These degrees are aggregated to obtain a degree of agreement of each pair of individuals as to their preferences between Q_1 (a linguistic quantifier, such as, “*most*”, “*almost all*”) pairs of relevant alternatives;
- The degrees of agreement resulting from step (2) are aggregated to obtain a degree of agreement of Q_2 (a linguistic quantifier similar to Q_1) pairs of important individuals as to their preferences between Q_1 pairs of relevant alternatives. This is the degree of consensus we seek.

Herrera-Viedma *et al* [38] proposed a group decision making framework which is different from the Kacprzyk-Fedrizzi-Nurmi’s scheme. Basically, this framework consists of four main processes:

1. *Resolution process* The goal of resolution process is to obtain a uniform representation of the preferences. Once the information is uniformed, a set of m individual fuzzy preference relations $\{P^1, \dots, P^m\}$ are available.
2. *Selection process* Selection process consists of two phases-*aggregation* and *exploitation*. The aggregation phase is to obtain a group collective preference relation $P^c = (p_{ij}^c)$ by means of the operator to aggregate all individual fuzzy preference relations $\{P^1, \dots, P^m\}$ and indicates the global preference between every ordered pair of alternatives according to the majority of experts’ opinions. Here, the aggregation operation is carried out by the linguistic quantifier guided OWA operator $\phi_Q(\cdot)$ [97][96].

In the exploitation phase, the group collective preference about the alternatives is transformed into a collective ranking of them, then a set of solution alternatives is obtained. The collective ranking is based on two choice degrees of alternatives: *the quantifier guided dominance degree (QGDD)* and *the quantifier guided nondominance degree (QGNDD)*, in which

$$QGDD_i = \phi_Q(p_{ij}^c | j = 1, \dots, n) \quad (25)$$

is used to quantify the dominance that alternative x_i is preferred by the whole group over Q (say, *most*) other alternatives, and

$$QGND D_i = \phi_Q (1 - p_{ji}^s | j = 1, \dots, n) \quad (26)$$

with $p_{ji}^s = \max(p_{ji}^c - p_{ij}^c, 0)$ is to represent the degree in which each alternative is not dominated by Q (say, *most*) other alternatives.

Finally, in terms of these two choice degrees, the solution X_{sol} is obtained by selecting those alternatives with maximum choice degrees.

3. *Consensus process* This consensus model is based on two consensus criteria: consensus degree and proximity degree. Both consensus criteria are defined by comparing the *position of alternative* in experts' individual solutions with the collective solution.
4. *Feedback process* A feedback mechanism acting as a guidance advisor is to substitute the roles of moderator in the traditional model of traditional group decision making. The proximity measures $(p_i(x_j), P_X^i)$ is used to build a feedback mechanism which helps experts change opinions for the sake of getting closer opinions between them, i.e., being helpful to reaching a higher degree of consensus decision. The feedback mechanism happens when the consensus level is not satisfactory.

Recently, Kwok *et al* proposed a fuzzy group decision making model with a structured decision making process [54], in which the group interaction was taken into account. Szmidt and Kacprzyk studied a consensus reaching process under intuitionistic fuzzy preference relations in [85], in which Spillman-Spillman-Bezdek approach to evaluation of the consensus of a group [84] was extended to the case when the preferences are expressed via intuitionistic fuzzy sets. The research results indicate that the intuitionistic fuzzy preference relations including a hesitation margin (concerning the membership degree) can better reflect the very imprecision of testimonies (expressing preferences) of the individuals during reaching a consensus [85]. In [40], Huynh and Nakamori proposed a satisfactory oriented approach to group decision making under linguistic assessments by using the named random preferences and probability theory, in which a linguistic choice function to establish a rank ordering among the alternatives was developed. By minimizing a sum of weighted dissimilarity among aggregated consensus and individual opinions, a method of aggregating individual fuzzy opinions into an optimal group consensus was suggested in [57]. An approach to consensus reaching for group decision with linguistic preference values was proposed by Mich *et al* [66]. In this approach, an opinion changing aversion function was defined for each expert to represent his/her resistance to opinion changing, the measure of consensus varies along with each expert's aversion to opinion change.

It is worth noting that in order to deal with random uncertainties exhibited in group decision making, some researchers have made efforts to build consensus models based on statistics theory and information theory, such as the research in [71][72][87][92], in which some degrees of consensus and aggregation functions are suggested from the viewpoints of probability, statistics and information theories.

V. Future Research

In group decision making under uncertain environments, given the range of opinions obtained from the panel of experts, reaching a consensus decision by taking into account these variations is a very challenging task. A promising framework is to use type-2 fuzzy sets to model the uncertainty exhibiting in the multiple experts consensus modelling process, in which ranking fuzzy sets is a necessary step [56][17][90]. However, ranking type-2 fuzzy sets for type-2 fuzzy consensus models is a much more difficult task. In [68], Mitchell proposed a method for ranking type-2 fuzzy numbers by interpreting each type-2 fuzzy number as weighted ensemble of ordinary (embedded) fuzzy numbers, and a complexity reduction technique was developed, but this method has not been verified for decision making tasks.

In applying type-2 fuzzy sets to decision making, parsimonious modelling methods by complexity reduction techniques and others are in great demand. Liang and Mendel originally suggested a method of designing parsimonious type-2 fuzzy model by using the SVD-QR with column pivoting algorithm to perform rule reduction [60]. However, some research on type-1 fuzzy models [111] indicates that the rule reduction by the SVD-QR with column pivoting algorithm heavily depends on the estimation of an effective rank. The problem is that different estimates of the rank often produce dramatically different rule reduction results. Hence, effective complexity reduction techniques including rule ranking methods are worthwhile to be investigated in the future.

In group decision making, aggregation is a necessary step to combine individual experts' preference or criteria into an overall one. A promising framework of aggregating linguistic information in group decision making is to use the aggregation operators that are able to aggregate the uncertain information with uncertain weights, such as the type-1 OWA operators [113]. Particularly, interval analysis techniques has been adapted to fuzzy interval computation where ending-points of intervals are changed into left and right fuzzy bounds, so the α -cut based interval analysis techniques would also possess the great potentials of being applied to improving the efficiency of type-1 OWA operation for real-time decision making.

VI. Conclusions

In group decision making, the group community shares responsibility for the decision and its outcome, one person does not have total responsibility for making a decision. The benefits of considering uncertainty measures in achieving consensus decision are not only to yield better solutions to the situations due to more inputs involved, but also to promote the growth of community and trust. However, it may often take a long time to reach a group consensus. In this paper, we have reviewed and analysed the state-of-the-art research efforts on group decision making from the perspective of linguistic uncertainty modelled by fuzzy sets, and some potential new research directions on fuzzy group decision making have been highlighted.

VII. Acknowledgment

This work has been supported by the EPSRC Research Grant EP/C542215/1.

References

- [1] K. J. Arrow, *Social Choice and Individual Values*, 2nd ed, New York: Wiley, 1963.
- [2] H. Bandemer and S. Gottwald, *Fuzzy Sets, Fuzzy Logic, Fuzzy Methods with Applications*, John Wiley & Sons, 1995.
- [3] G. Beliakov and J. Warren, Appropriate choice of aggregation operators in fuzzy decision support systems, *IEEE Trans. on Fuzzy Systems*, 9(6), pp.773-784, 2001.
- [4] G. Beliakov, How to build aggregation operators from data, *International Journal of Intelligent Systems*, 18, pp.903-923, 2003.
- [5] R. E. Bellman and L. A. Zadeh, Decision making in a fuzzy environment, *Management Science*, 7(4), pp.141-164, 1970.
- [6] D. Ben-Arieh and Z. Chen, Linguistic-labels aggregation and consensus measure for autocratic decision making using group recommendations, *IEEE Trans. on Systems, Man and Cybernetics-part A*, 36(3), pp.558-568, 2006.
- [7] J. C. Bezdek, B. Spillman, and R. Spillman, Fuzzy relation spaces for group decision theory: An application, *Fuzzy Sets and Systems*, 4(1), pp.5-14, 1979.
- [8] P. P. Bonissone and K. S. Decker, Selecting uncertainty calculi and granularity: An experiment in trading-off Precision and complexity, in L. N. Kanal, J. F. Lemmer (eds.), *Uncertainty in Artificial Intelligence*, (North Holland), pp. 217-247, 1986.
- [9] G. Bordogna, M. Fedrizzi, and G. Pasi, A linguistic modeling of consensus in group decision making based on OWA operators, *IEEE Trans. on Systems, Man and Cybernetics-part A*, 27(1), pp.126-133, 1997.
- [10] C. E. Bozdog, C. Kahraman and D. Ruan, Fuzzy group decision making for selection among computer integrated manufacturing systems, *Computers in Industry*, 51(1), pp.13-29, 2003.
- [11] J. Buckley, *Fuzzy Probability and Statistics*, Springer Verlag, 2006.
- [12] J. L. Chaneau, M. Gunaratne, and A. G. Altschaeffl, An application of type-2 sets to decision making in engineering, *Analysis of Fuzzy Information-Vol. II: Artificial Intelligence and Decision Systems*, J. C. Bezdek, Ed. Boca Raton, FL: CRC, 1987.
- [13] S. J. Chen and C. L. Hwang, *Fuzzy Multiple Attribute Decision Making: Methods and Applications*. New York: Springer-Verlag, 1992.
- [14] C.-H. Cheng, A simple fuzzy group decision making method, *Proc. of FUZZ-IEEE*, 22-25 August, Seoul, Korea, pp.910-915, 1999.
- [15] F. Chiclana, F. Herrera, and E. Herrera-Viedma, Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations, *Fuzzy Sets and Systems*, 97, pp.33-48, 1998.
- [16] F. Chiclana, F. Herrera, and E. Herrera-Viedma, Integrating multiplicative preference relations in a multipurpose decision making model based on fuzzy preference relations, *Fuzzy Sets and Systems*, 112, pp.277-291, 2001.
- [17] T.-C. Chu and C. T. Tsao, Ranking fuzzy numbers with an area between the centroid point and original point, *Computers and Mathematics with Applications*, 43, pp.111-117, 2002.
- [18] V. Cutello and J. Montero, Hierarchies of aggregation operators, *International Journal of Intelligent Systems*, 9, pp.1025-1045, 1994.
- [19] V. Cutello, and J. Montero, Hierarchies of intensity preference aggregations, *International Journal of Approximate Reasoning*, 10(2), pp.123-133, 1994, .
- [20] J.M. da Costa Sousa and U. Kaymak, *Fuzzy Decision Making in Modeling and Control*. World Scientific, 2002.
- [21] O. Despic and S. P. Simonovic, Aggregation operators for soft decision making in water resources, *Fuzzy Sets and Systems*, 115, pp.11-33, 2000.

- [22] W. M. Dong and F.S. Wong, Fuzzy weighted averages and implementation of the extension principle, *Fuzzy Sets and Systems*, 21, pp.183-199, 1987.
- [23] D. Dubois and J-L. Koning, Social choice axioms for fuzzy set aggregation, *Fuzzy Sets and Systems*, 43, pp.257-274, 1991.
- [24] D. Dubois and H. Prade, A review of fuzzy set aggregation connectives, *Information Science*, 36, pp.85-121, 1985.
- [25] D. Dubois and H. Prade, Fuzzy sets and probability: misunderstandings, bridges and gaps, *Proc. of the FUZZ-IEEE, San Francisco, CA, 28 March 1 April*, pp.1059-1068, 1993.
- [26] M. Fedrizzi, J. Kacprzyk and S. Zadrozny, An interactive multiuser decision support system for consensus reaching processes using fuzzy logic with linguistic quantifiers, *Decision Support Systems*, 4, pp.313-327, 1988.
- [27] M. Fedrizzi and J. Kacprzyk, On measuring consensus in the setting of fuzzy preference relations, in K. Kacprzyk and M. Roubens (eds), *Non-Conventional Preference Relations in Decision Making*, New York: Springer-Verlag, pp.129-141, 1988.
- [28] M. Fedrizzi and J. Kacprzyk, Consensus degrees under fuzzy majorities and preferences using ordered weighted average operators, *Proc. of the 5th IFSA World Congress, Seoul, Korea*, pp.624-626, 1993.
- [29] J. Fodor and M. Roubens, *Fuzzy Preference Modeling and Multicriteria Decision Support*, Kluwer, 1994.
- [30] D. Filev and R. R. Yager, Learning OWA operator weights from data, *Proc. of 3rd FUZZ-IEEE Conference, Orlando, FL, USA*, pp.468-473, 1994.
- [31] D. Filev and R. R. Yager, On the issue of obtaining OWA operator weights, *Fuzzy Sets and Systems*, 94, pp.157-169, 1998.
- [32] D. Filev and R. R. Yager, Analytic properties of maximum entropy OWA operators, *Information Sciences*, 85, pp.11-27, 1995.
- [33] R. Fuller and C. Carlsson, Fuzzy multiple criteria decision making: recent developments, *Fuzzy Sets and Systems*, 78, pp.139-153, 1996.
- [34] R. Fuller and P. Majlender, An analytic approach for obtaining maximal entropy OWA operator weights, *Fuzzy Sets and Systems*, 124(1), pp. 53-57, 2001.
- [35] M. Grabisch, S. A. Orlovski, and R. R. Yager, Fuzzy aggregations of numerical preferences, in R. Slowinski (ed), *Fuzzy Sets in Decision Analysis, Operations Research and Statistics*, Kluwer Academic Publishers, 1998.
- [36] H. A. Hagrass, A hierarchical type-2 fuzzy logic control architecture for autonomous mobile robots, *IEEE Trans. on Fuzzy Systems*, 12(4), pp.524-539, 2004.
- [37] F. Herrera and E. Herrera-Viedma, Linguistic decision analysis: steps for solving decision problems under linguistic information, *Fuzzy Sets and Systems*, 115, pp.67-82, 2000.
- [38] E. Herrera-Viedma, F. Herrera and F. Chiclana, A consensus model for multiperson decision making with different preference structures, *IEEE Trans. on Systems, Man and Cybernetics-part A*, 32(3), pp.394-402, 2002.
- [39] E. Herrera-Viedma, F. Chiclana, F. Herrera, S. Alonso, Group decision-making model with incomplete fuzzy preference relations based on additive consistency, *IEEE Trans. on Systems, Man and Cybernetics-part B*, 37(1), pp.176-189, 2007.
- [40] V.-N. Huynh and Y. Nakamori, A satisfactory-oriented approach to multiexpert decision-making with linguistic assessments, *IEEE Trans. on Systems, Man and Cybernetics-part B*, 35(2), pp.184-196, 2005.
- [41] R. I. John, P. R. Innocent and M. R. Barnes, Neuro-fuzzy clustering of radiographic tibia image data using type 2 fuzzy sets, *Information Sciences*, 125, pp.65-82, 2000.
- [42] R. I. John and P. R. Innocent, Modeling uncertainty in clinical diagnosis using fuzzy logic, *IEEE Trans. on Systems, Man and Cybernetics-part B*, 35(6), pp.1340-1350, 2005.
- [43] J. Kacprzyk, Group decision making with a fuzzy linguistic majority, *Fuzzy Sets and Systems*, 18, pp.105-118, 1986.
- [44] J. Kacprzyk, On some fuzzy cores and "soft" consensus measures in group decision making, in J. C. Bezdek (ed.), *The Analysis of Fuzzy Information*, 2, CRC Press, pp.119-130, 1987.
- [45] J. Kacprzyk and M. Fedrizzi, A "soft" measure of consensus in the setting of partial (fuzzy) preferences, *Europ. Journal of Operational Research*, 34, pp.315-325, 1988.
- [46] J. Kacprzyk and H. Nurmi, Group decision making under fuzziness, in R. Slowinski (ed), *Fuzzy Sets in*

- Decision Analysis, Operations Research and Statistics, Kluwer Academic Publishers, 1998.
- [47] J. Kacprzyk, M. Fedrizzi, and H. Nurmi, Group decision making and consensus under fuzzy preferences and fuzzy majority, *Fuzzy Sets and Systems*, 49, pp.21-31, 1992.
- [48] N. N. Karnik and J. M. Mendel, Introduction to type-2 fuzzy logic systems, *Proc. of FUZZ-IEEE Conference*, Anchorage, AK, pp.915-920, 1998.
- [49] N. N. Karnik, J. M. Mendel, and Q. Liang, Type-2 fuzzy logic systems, *IEEE Trans. on Fuzzy Systems*, 7(6), pp.643-658, 1999.
- [50] G.J. Klir and T.A. Folger, *Fuzzy Sets, Uncertainty and Information*. Singapore: Prentice-Hall, 1992.
- [51] B. Kosko, Fuzziness vs. probability, *International Journal of General Systems*, 17, pp.211-240, 1990.
- [52] V. Kreinovich and H. T. Nguyen, 1st order, 2nd order, what next? do we really need third-order descriptions: a view from a realistic (granular) viewpoint, *Proceedings of the Joint 9th World Congress of the International Fuzzy Systems Association and 20th International Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS*, Vancouver, Canada, July 25-28, 2001, pp. 1908-1913.
- [53] L. I. Kuncheva, Using degree of consensus in two-level fuzzy pattern recognition, *European Journal of Operational Research*, 80, pp.365-370, 1995.
- [54] R. C. W. Kwok, J. Ma and D. Zhou, Improving group decision making: a fuzzy GSS approach, *IEEE Trans. on Systems, Man and Cybernetics-part C*, 32(1), pp.54-62, 2002.
- [55] H.-S. Lee, Aggregation of fuzzy opinions under group decision making environment, *Proc. of the 10th IEEE International Fuzzy Systems Conference (Fuzz-IEEE)*, 2-5 December, Melbourne, Australia, pp.172-175, 2001.
- [56] H. Lee-Kwang and J. -H. Lee, A method for ranking fuzzy numbers and its application to decision, *IEEE Trans. on Fuzzy Systems*, 7(6), pp.677-685, 1999.
- [57] H.-S. Lee, Optimal consensus of fuzzy opinions under group decision making environment, *Fuzzy Sets and Systems*, 132(3), pp.303-315, 2002.
- [58] H.-X. Li and V. C. Chen, *Fuzzy Sets and Fuzzy Decision Making*, CRC press, 1995.
- [59] L. Li, W.-H. Lin and H. Liu, Type-2 fuzzy logic approach for short-term traffic forecasting, *IEE Proc. Intell. Transp. Syst.*, 153(1), pp.33-40, 2006.
- [60] Q. Liang and J. M. Mendel, Designing interval type-2 fuzzy logic systems using an SVD-QR method: rule reduction, *International Journal of Intelligent Systems*, 15, pp.939-957, 2000.
- [61] M. U. Marimin, I. Hatono and H. Tamura, Linguistic labels for expressing fuzzy preference relations in fuzzy group decision making, *IEEE Trans. on Systems, Man, and Cybernetics-Part B*, 28(2), pp.205-218, 1998.
- [62] J. M. Mendel, Uncertainty, fuzzy logic, and signal processing, *Signal Processing*, 80(6), pp. 913-933, 2000.
- [63] J. M. Mendel, *Uncertainty Rule-based Fuzzy Logic Systems: Introduction and New Directions*, Prentice Hall, Inc., 2001.
- [64] J. M. Mendel and R. I. John, Type-2 fuzzy sets made simple, *IEEE Trans. on Fuzzy Systems*, 10(2), pp.117-127, 2002.
- [65] J. M. Mendel, R. I. John, and F. Liu, Interval type-2 fuzzy logic systems made simple, *IEEE Trans. on Fuzzy Systems*, 14(6), pp.808-821, 2006.
- [66] L. Mich, M. Fedrizzi, and L. Gaio, Approximate reasoning in the modeling of consensus in group decisions, in E. P. Klement and W. Slany (eds.), *Fuzzy Logic in Artificial Intelligence*, Berlin, Germany: Springer-Verlag, pp.91-102, 1993.
- [67] H. B. Mitchell, Pattern recognition using type-II fuzzy sets, *Information Sciences*, 170, pp.409-418, 2005.
- [68] H. B. Mitchell, Ranking type-2 fuzzy numbers, *IEEE Trans. on Fuzzy Systems*, 14(2), pp.287-294, 2006 .
- [69] H. B. Mitchell and D. D. Estrakh, A modified OWA operator and its use in lossless DPCM image compression, *International Journal of Uncertain Fuzziness, Knowledge Based Systems*, 5, pp.429-436, 1997.
- [70] B. Möller, W. Graf, M. Beer and J.-U. Sickert, Fuzzy randomness - towards a new modeling of uncertainty, *Proc. of Fifth World Congress on Computational Mechanics*, 7-12 July, 2002, Vienna, Austria.
- [71] M. Mostaghimi, Bayesian estimation of a decision using information theory, *IEEE Trans. on Systems, Man and Cybernetics-Part A*, 27(4), pp.506-517, 1997.
- [72] K.-C. Ng and B. Abramson, Consensus diagnosis: a simulation study, *IEEE Trans. on Systems, Man and Cybernetics*, 22(5), pp.916-928, 1992.

- [73] M. O'Hagan, Aggregating template or rule antecedents in real-time expert systems with fuzzy set logic, Proc. of the 22nd Annual IEEE Asilomar Conf. on Signals, Systems and Computers, Pacific Grove, CA, USA, pp.681-689, 1988.
- [74] S. A. Orlovsky, Decision-making with a fuzzy preference relation, Fuzzy Sets and Systems, 1, pp.155-167, 1978.
- [75] T. Ozen and J. M. Garibaldi, Effect of type-2 fuzzy membership function shape on modelling variation in human decision making, Proc. of FUZZ-IEEE, 25-29 July, Budapest, Hungary, pp. 971-976, 2004.
- [76] T. Ozen, J. M. Garibaldi, and S. Musikasuwana, Modelling the variation in human decision making, Proc. of IEEE Annual Meeting of Fuzzy Information Processing NAFIPS'04, Banff, Canada, 27-30 June, pp.617-622, 2004.
- [77] T. Ozen, J.M. Garibaldi, and S. Musikasuwana, Preliminary investigations into modelling the variation in human decision making, Proc. of Information Processing and Management of Uncertainty in Knowledge Based Systems (IPMU), Perugia, Italy, 4-9 July, 2004.
- [78] J. I. Peláez, J.M. Doña and D. La Red (). "Analysis of the majority process in group decision making process, Proc. of the 7th Joint Conference on Information Sciences, 26-30 September, North Carolina, USA, pp.155-159, 2003.
- [79] J. I. Peláez and J.M. Doña, Majority additive-ordered weighting averaging: a new neat ordered weighting averaging operator based on the majority process, International Journal of Intelligent Systems, 18, pp.469-481, 2003.
- [80] P. Prodanovic and S. P. Simonovic, Fuzzy compromise programming for group decision making, IEEE Trans. on Systems, Man and Cybernetics-Part A, 33(3), pp.358-365, 2003.
- [81] D. A. Ralescu, Fuzzy random variables revisited, Proc. IFES '95 and Fuzzy IEEE Joint Conf., 2, Yokohama, Japan), pp.993-1000, 1995.
- [82] Th. L. Saaty, The Analytic Hierarchy Process, New York: McGraw-Hill, 1980.
- [83] F. Seo and M. Sakawa, Fuzzy multiattribute utility analysis for collective choice, IEEE Trans. on Systems, Man and Cybernetics, 15, pp.45-53, 1985.
- [84] B. Spillman, R. Spillman, J. Bezdek, A fuzzy analysis of consensus in small groups, in P. Wang, S. Chang (eds), Fuzzy Sets Theory and Applications to Policy Analysis and Information Systems. New York: Plenum Press, pp.291-308, 1980.
- [85] E. Szmidt, and J. Kacprzyk, A consensus-reaching process under intuitionistic fuzzy preference relations, International Journal of Intelligent Systems, 18, pp.837-852, 2003.
- [86] T. Tanino, On group decision making under fuzzy preferences, in J. Kacprzyk and M. Fedrizzi (eds), Multiperson Decision Making Using Fuzzy Sets and Possibility Theory, Norwell, MA: Kluwer, pp.172-185, 1990.
- [87] W. J. Tastle and M. J. Wierman, Consensus and dissent: a new measure of agreement, Proc. of NAFIPS, pp.385-388, 2005.
- [88] V. Torra, On the learning of weights in some aggregation operators: the weighted mean and OWA operators, Mathware Soft Computing, 6, pp.249-265, 1999.
- [89] V. Torra, OWA operators in data modeling and re-identification, IEEE Trans. on Fuzzy Systems, 12(5), pp.652-660, 2004.
- [90] T.-Y. Tseng and C. M. Klein, New algorithm for the ranking procedure in fuzzy decision-making, IEEE Trans. on Systems, Man and Cybernetics, 19(5), pp.1289-1296, 1989.
- [91] P. Z. Wang, Fuzzy Sets and Falling Shadows of Random Sets, Beijing Normal University Press, 1985.
- [92] M. J. Wierman and W. J. Tastle, Consensus and dissent: theory and properties, Proc. of NAFIPS, pp.385-388, 2005.
- [93] D. Wu and J. Mendel, The linguistic weighted average, Proc. of FUZZ-IEEE, 16-21 July, Vancouver, BC, Canada, pp.3030- 3037, 2006.
- [94] Z. S. Xu and Q. L. Da, An overview of operators for aggregating information, International Journal of Intelligent Systems, 18, pp.953-969, 2003.
- [95] R. R. Yager, Fuzzy subsets of type II in decisions, Journal of Cybernetics, 10, pp. 137-159, 1980.
- [96] R. R. Yager, Families of OWA operators, Fuzzy Sets and Systems, 59, pp.125-148, 1993.
- [97] R. R. Yager, On ordered weighted averaging aggregation operators in multi-criteria decision making, IEEE Trans. on Systems, Man and Cybernetics, 18, pp.183-190, 1988.
- [98] R. R. Yager, A general approach to criteria aggregation using fuzzy measures, International Journal of Man-Machine Study, 38, pp.187-213, 1993.

- [99] R. R. Yager, "The induced fuzzy integral aggregation operator, *International Journal of Intelligent Systems*, 17, pp.1049-1065, 2002.
- [100] R. R. Yager, Induced aggregation operators, *Fuzzy Sets and Systems*, 103, pp.59-69, 2003.
- [101] R. R. Yager and D. P. Filev, *Essentials of Fuzzy Modeling and Control*, John Wiley & Sons, New York, 1994.
- [102] R. R. Yager and D. P. Filev, Parameterized and-like and or-like OWA operators, *International Journal of General Systems*, 22, pp.297-316, 1994.
- [103] R. R. Yager and D. P. Filev, Generalizing the modeling of fuzzy logic controllers by parameterized aggregation operators, *Fuzzy Sets and Systems*, 70, pp.303-313, 1995.
- [104] R. R. Yager, D. P. Filev, and T. Sadeghi, Analysis of flexible structured fuzzy logic controllers, *IEEE Trans. on Systems, Man and Cybernetics*, 24, pp.1035-1043, 1994.
- [105] R. R. Yager, L. S. Goldstein, E. Mendels, FUZZ-MAR: an approach to aggregating market research data based on fuzzy reasoning, *Fuzzy Sets and Systems*, 68(1), pp.1-11, 1994.
- [106] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8, pp.338-353, 1965.
- [107] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-1, *Information Science*, 8, pp. 199-249, 1975.
- [108] L. A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, *Comput Math Appl*, 9, pp.149-184, 1983.
- [109] L. A. Zadeh, From computing with numbers to computing with words - from manipulation of measurements to manipulation of perceptions, *IEEE Trans. on Circuits and Systems*, 45(1), pp.105-119, 1999.
- [110] J. Zeng and Z. Q. Liu, Type-2 fuzzy sets for handling uncertainty in pattern recognition, *Proc. FUZZ-IEEE*, July, Vancouver, Canada, pp.6597-6602, 2006.
- [111] S.-M. Zhou, and J. Q. Gan, Constructing parsimonious fuzzy classifiers based on L2-SVM in high-dimensional space with automatic model selection and fuzzy rule ranking, *IEEE Trans. on Fuzzy Systems*, 15(3), pp.398-409, 2007.
- [112] S.-M. Zhou, J. M. Garibaldi, R. I. John, and F. Chiclana, On constructing parsimonious type-2 fuzzy logic systems via influential rule selection, *IEEE Trans. on Fuzzy Systems*(accepted), 2008.
- [113] S.-M. Zhou, F. Chiclana, R. I. John and J. M. Garibaldi, Type-1 OWA operators for aggregating uncertain information with uncertain weights induced by type-2 linguistic quantifiers, *Fuzzy Sets and Systems* (accepted), 2008.
- [114] H. -J. Zimmermann, *Fuzzy Sets, Decision Making and Expert Systems*, Kluwer Academic Publishers, Boston, 1987.
- [115] H. -J. Zimmermann and P. Zysno, Latent connectives in human decision making, *Fuzzy Sets and Systems*, 4, pp.37-51, 1980.