Constrained Type-2 Fuzzy Sets

Jonathan M. Garibaldi
Intelligent Analysis and Modelling (IMA) Research Group
School of Computer Science, University of Nottingham
Jubilee Campus, Nottingham, UK, NG8 1BB.
Email: jmg@cs.nott.ac.uk

Sergio Guadarrama
Berkeley Initiative in Soft Computing (BISC)
EECS Department
University of California at Berkeley
Email: sguada@eecs.berkeley.edu

***Abstract*—Type-2 fuzzy sets extend the expressive capabilities of type-1 fuzzy sets in that, in addition to represent imprecise concepts, they are able to represent the imprecision in the membership function of fuzzy sets. Their use is particularly appropriate when modelling linguistic concepts such as words that mean slightly different things to different people. However, type-2 fuzzy sets do not place any constraints upon the continuity and other properties of their embedded sets. We argue that, *for some concepts*, additional properties constraining both the allowable footprint of uncertainty and the embedded sets of type-2 sets are desirable. We present a novel formulation of a *constrained type-2 fuzzy set* that achieves this, and then detail some of the properties of this novel form of type-2 fuzzy sets.**

***Keywords*—Constrained Fuzzy Sets, Type-2 Fuzzy Sets, Constrained Type-2 Fuzzy Sets**

I. Introduction

Fuzzy sets were introduced by Zadeh in order to model imprecise sets that "play an important role in human reasoning" [1]. Later, Zadeh clarified the definition of the conventional fuzzy sets as those taking a real value in $[0, 1]$ at each value of the domain: these are termed 'type-1' fuzzy sets [2]. As an example, consider a standard type-1 fuzzy set corresponding to a meaning of *medium* height of a person (where the domain of height ranges between 1 and 2 metres): an example of such a set is given in Figure 1.

It is clear that, while a type-1 fuzzy set extends the notion of membership of classical (crisp) sets by allowing the membership of a set to be a real number between 0 and 1, it is still in some sense 'precise', in that for any given value of height (for example 1.8m) the membership of the set of medium height persons is a precise real number ($\mu = 0.37$, in this example). The fact that the same concept (*medium* height) means different things to different people has been clearly demonstrated by Mendel [3] and Guadarrama [4]. So, while one person may consider a height of 1.8m to be medium to degree 0.37, another may consider it to be (say) 0.40.

Zadeh had previously recognised this point and had outlined the definition of "fuzzy sets with fuzzy membership functions" which he called type-n, $n = 2, 3, \dots,$ in which the membership values range over fuzzy sets of type $n - 1$ [2] — so, for example, fuzzy sets of type-2 have membership values which are type-1 fuzzy sets. Thus, type-2 fuzzy sets have an extra 'dimension', which has an unfortunate corollary of making them hard to represent visually. Type-2 fuzzy sets can be visually simplified through a concept known as the 'footprint of uncertainty (FOU)' introduced by Mendel [3] which captures the bounded regions that have secondary degrees greater than zero. The FOU can then be visualised by shading this bounded region, as shown in the possible FOU in Figure 2 for the *medium* height example presented in Figure 1. In this example, for a height of 1.8m, the membership of the *medium* set now lies between 0.11 and 0.78.

In a (general) type-2 fuzzy set, the membership is actually a fuzzy set ranging over the domain $[0, 1]$. Mendel (and others) have made a distinction between *general* type-2 fuzzy sets,

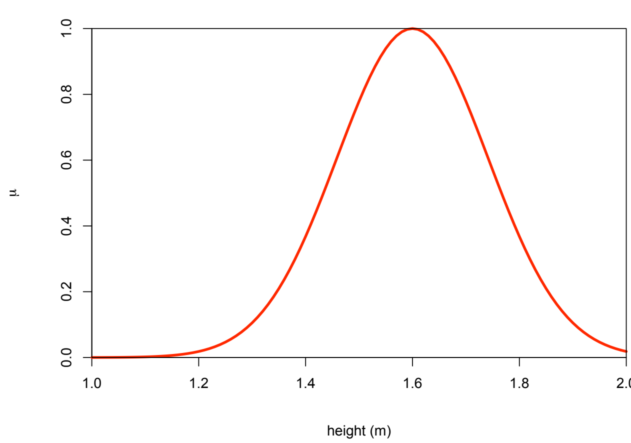

Fig. 1. A possible type-1 fuzzy set for modelling the concept of *medium* height (in metres).

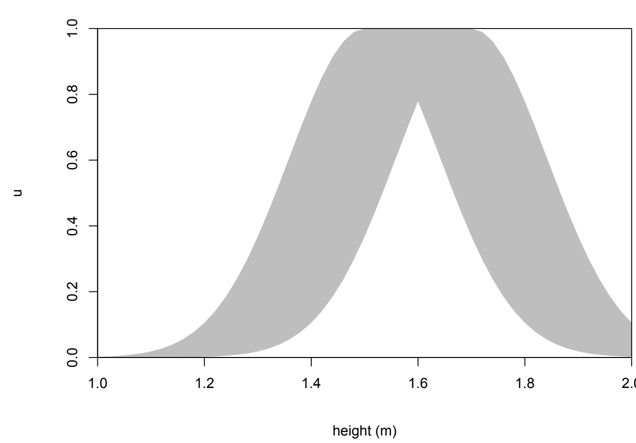

Fig. 2. A possible footprint of uncertainty for a type-2 fuzzy set corresponding to the same concept of *medium* height (in metres) as shown in Figure 1.

in which these membership values may themselves range between 0 and 1 at each point (in which case, this 'extra dimension' must be maintained in all representations and calculations), and *interval* type-2 fuzzy sets in which the values are either zero or one, in which case the FOU fully characterises the type-2 fuzzy sets and the 'extra dimension' may be dropped.

In this paper, we argue that the original generic definitions of type-2 fuzzy sets provided by Zadeh are too generic *when considered in certain contexts*, in that they do not impose any conditions on the membership functions which can be used. In particular, the functions do not have to be continuous, monotonic, etc. While such conditions are straight forward to impose on type-1 membership functions, in the case of type-2 membership functions they are not. The layout of the paper is as follows. In the next Section, we present some standard concepts of type-2 fuzzy sets, in order to define the terms and notations that are used in the paper. In Section III, we outline the semantic difficulties which arise from the standard definitions, and in Section IV, we present our novel formulation of type-2 fuzzy sets which we believe overcome these difficulties. In Section V, we define some essential operations on these new sets, and outline the inferencing, type-reduction and defuzzification processes. In Section VI we discuss some of the implications of our proposals, and then we conclude the paper with a brief summary and an outline of future work.

II. Definitions and Notation

In order to illustrate a property of type-2 fuzzy sets which we consider to cause difficulties of semantic interpretation, we firstly present some essential concepts and notations.

Definition 1: A *type-1 fuzzy set*, denoted T1-FS, is characterised by a type-1 membership function

$$\mu_{\tilde{A}} : X \to \underset{U}{[0,1]} \qquad \blacksquare$$

In this case, X is referred to as the *primary domain* and U as the *primary membership* of x.

Definition 2: An *interval-type fuzzy set*, denoted IT-FS, is characterised by an interval membership function

$$\mu^I_{\tilde{A}} : X \to L(\underset{U}{[0,1]})$$

where $L([0,1]) = \{[a,b] \,|\, 0 \le a \le b \le 1\}$ is the set of all the closed subintervals of [0,1]. ■

In this case also, X is referred to as the *primary domain* and U as the *primary membership* of x.

Definition 3: A *type-2 fuzzy set*, denoted T2-FS, is characterised by a type-2 membership function

$$\tilde{\mu}_{\tilde{A}} : X \times \underset{U}{[0,1]} \to \underset{V}{[0,1]} \qquad \blacksquare$$

Nevertheless in this case, X is referred to as the *primary domain* and U as the *secondary domain* of the type-2 fuzzy set, and V as the *secondary membership* of x.

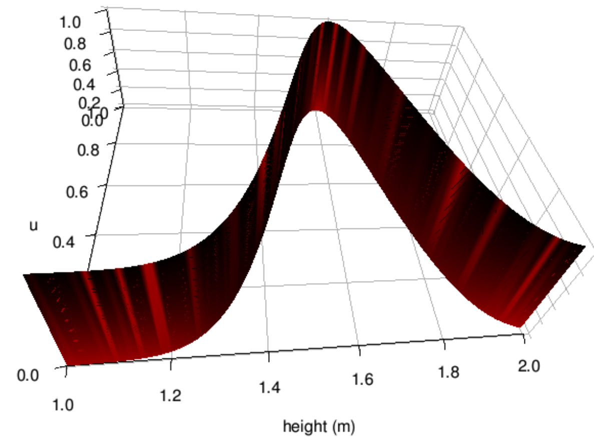

Fig. 3. A type-2 fuzzy set representing a type-1 fuzzy set.

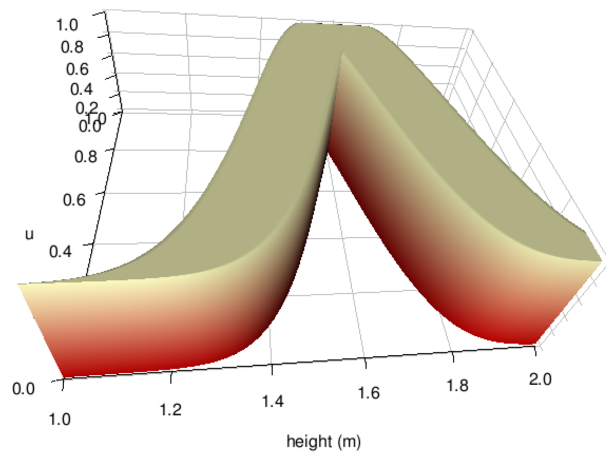

Fig. 4. A type-2 fuzzy representing an interval fuzzy set.

Remark 1: In our definitions we are going to denote any fuzzy set by $\tilde{A}$, regardless of whether it is represented by type-1, interval or type-2 membership functions. We are going to denote a type-1 membership function of the fuzzy set $\tilde{A}$ by $\mu_{\tilde{A}}$, an interval membership function by $\mu^I_{\tilde{A}}$, and a type-2 membership function by $\tilde{\mu}_{\tilde{A}}$. ■

Remark 2: Any T1-FS can be represented as a special type of T2-FS:

$$\tilde{\mu}_{\tilde{A}}(x,u) = \begin{cases} 1; u = \mu_{\tilde{A}}(x) \\ 0; \text{otherwise} \end{cases} \quad \text{(see Figure 3)}$$

Also, any IT-FS can be represented as a special type of T2-FS (usually called interval type-2 fuzzy set or IT2-FS):

$$\tilde{\mu}_{\tilde{A}}(x,u) = \begin{cases} 1; u \in \mu^I_{\tilde{A}}(x) \\ 0; \text{otherwise} \end{cases} \quad \text{(see Figure 4)} \qquad \blacksquare$$

Definition 4: An *embedded type-2 fuzzy set*, denoted ET2-FS, represents a path along a type-2 fuzzy set and contains only one primary degree u_x for each x, each with its associated secondary grade v_x, i.e.,

$$\tilde{\mu}_{\tilde{A}}(x, u_x) = v_x, \forall x \in X \qquad \blacksquare$$

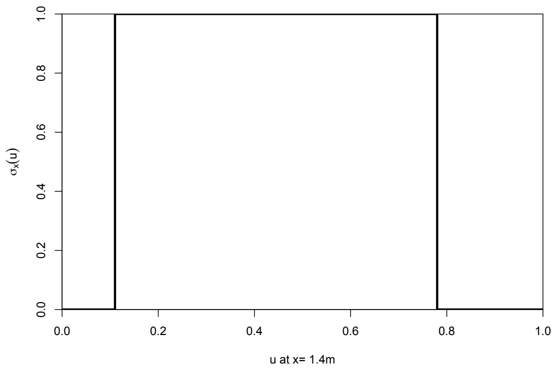

Fig. 5. The x-slice at $x = 1.4$m of the type-2 set shown in Figure 4.

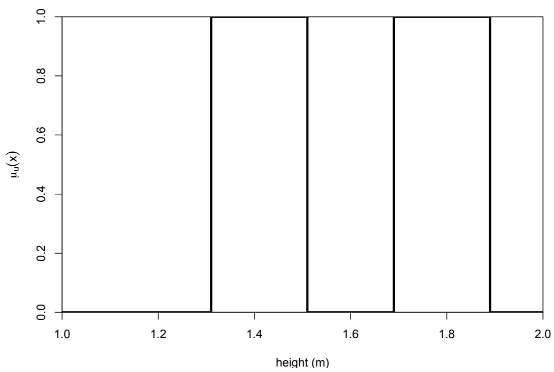

Fig. 6. The u-slice at $u = 0.4$ of the type-2 set shown in Figure 4.

Definition 5: An *embedded type-1 fuzzy set*, denoted ET1-FS, represents a projection of an ET2-FS, in which the secondary degree has been dropped, that is, it contains only one primary degree u_x for each x (see Figure 7):

$$\mu_{\tilde{A}}(x) = u_x; \tilde{\mu}_{\tilde{A}}(x, u_x) = v_x, \forall x \in X \qquad \blacksquare$$

Definition 6: A $x - slice$ (also known as vertical-slice) of a type-2 fuzzy set is a type-1 fuzzy set $\mu_x : [0,1] \rightarrow [0,1]$ defined between the secondary domain and the secondary degree, and built by slicing the type-2 fuzzy set by a vertical plane $x = x_0$, i.e. (see Figure 5):

$$\mu_{x_0}(u) = \tilde{\mu}_{\tilde{A}}(x = x_0, u) \qquad \blacksquare$$

Definition 7: A $u{-}slice$ (also known as horizontal-slice) of a type-2 fuzzy set is a type-1 fuzzy set $\mu_u : X \rightarrow [0,1]$ defined between the primary domain and the secondary degree, and built by slicing the type-2 fuzzy set by a horizontal plane $u = u_0$, i.e. (see Figure 6):

$$\mu_{u_0}(x) = \tilde{\mu}_{\tilde{A}}(x, u = u_0) \qquad \blacksquare$$

Definition 8: A $v - slice$ (also known as alpha-slice or z-slice) of a type-2 fuzzy set is an interval fuzzy set $\mu_v^I : X \rightarrow [0,1] \times [0,1]$ defined between the primary domain and the secondary domain, and built by slicing the type-2 fuzzy set by a transversal plane $v = v_0$ (see Figure 7):

$$\mu_{v_0}^I(x) = [a,b]; u \in [a,b], v_0 = \tilde{\mu}_{\tilde{A}}(x, u) \qquad \blacksquare$$

Definition 9: The *footprint of uncertainty* (FOU) of a T2-FS captures the bounded regions that have secondary degrees greater than zero, and can be represented by an IT-FS, as the limit $v - slice$ for $v > 0$ with $v \rightarrow 0$ (see Figure 2):

$$FOU(\tilde{\mu}_{\tilde{A}}) = \underset{v \rightarrow 0}{Lim}\ \mu_v^I(x) \qquad \blacksquare$$

Definition 10: A similarity relation $\delta : X \times X \rightarrow [0,1]$ is fuzzy relation [5] verifying the following properties:

Reflexive: $\delta(x, x) = 1; \forall x \in X$
Symmetric: $\delta(x, y) = \delta(y, x); \forall x, y \in X$
T-Transitive: $T(\delta(x, y), \delta(y, z)) \leq \delta(x, z); \forall x, y, z \in X$

for some t-norm T. $\blacksquare$

III. A Difficulty with Type-2 Fuzzy Sets

The essential difficulty with type-2 fuzzy sets can most easily be observed by considering possible type-1 embedded sets of the type-2 set with the FOU shown in Figure 2. One possible type-1 embedded set of this type-2 set might be as shown in red in Figure 7. This embedded set exhibits the same Gaussian form on the primary domain as the type-1 set shown in Figure 1, and may be thought of as this type-1 fuzzy set shifted to the left by a small amount.

However, the definition of a type-2 fuzzy set and the associated definition of embedded sets places no requirement on the continuity of $\tilde{\mu}_{\tilde{A}}(x, u_x)$. That is $\tilde{\mu}_{\tilde{A}}(x + \delta x, u_{x+\delta x})$ is entirely independent and unrelated to $\tilde{\mu}_{\tilde{A}}(x, u_x)$. Hence, the form of the embedded set is completely unrestricted, and the set shown in red in Figure 8 is perfectly acceptable as an embedded set of the type-2 fuzzy set modelling the concept of *medium height*, although discontinuous and non-convex. We refer to such embedded sets as *unconstrained*. However, it seems clear that such unconstrained embedded sets have little or no meaning from the perspective of modelling the human concept of *medium height*. That is, while such an embedded set is mathematically possible, it is essentially *meaningless* **in this particular context**. Note that this lack of meaning is heavily dependent on the context — it is perfectly possible that a non-convex embedded set (for example) may be acceptable

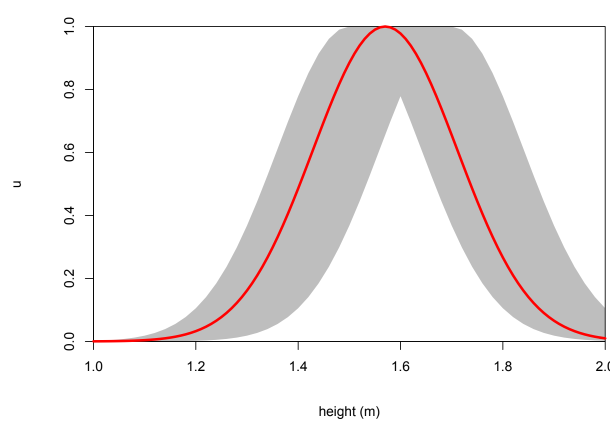

Fig. 7. A continuous type-1 embedded set within the FOU of a T2-FS.

in a human reasoning context and it is perfectly possible that any arbitrary embedded set *is meaningful* in a context other than modelling human reasoning (for example, in a type-2 fuzzy control context).

Does the presence within a type-2 fuzzy set of unconstrained embedded sets actually matter or make any difference? We believe is is important for two main reasons:

1) The unconstrained nature of the associated embedded sets causes an essential *semantic difficulty* with the interpretation of a type-2 fuzzy set. It is surely the case that when modelling a concept such as *tall (height)* that, for any given observer at any given time, the *tallness* of someone 1.96m tall is greater than the *tallness* of someone 1.95m tall, i.e. $\mu_{\text{tall}}(1.96) \geq \mu_{\text{tall}}(1.95)$, whether or not the μ is type-1 or type-2. While a type-2 set approach may be employed to represent uncertainty in the membership function of a fuzzy set, it should be possible to maintain this requirement somehow.
2) The existing formulation of type-2 sets allows too much freedom in their mathematical properties which will result in 'inappropriate' answers. The unconstrained embedded sets of conventional type-2 sets are maintained and processed through the type-2 inferencing process. Consider, for example, the usual Karnik-Mendel method of type reduction [6]. The method explicitly requires evaluation of all the embedded sets in order to determine the leftmost centroid and the rightmost centroid. But, if many of the unconstrained embedded sets are not meaningful in the given context, why inference with them and why evaluate them all during type-reduction? If the embedded sets are constrained, and these constraints are maintained through inferencing, then different results will be obtained.

Of course, a conventional definition of a type-1 fuzzy set places no requirement on continuity, monotonicity, convexity, etc., of the membership function, but in the type-1 situation, such properties are trivial to impose.

Furthermore, it is often (although not always) the case that additional *semantic constraints* are imposed on the member-

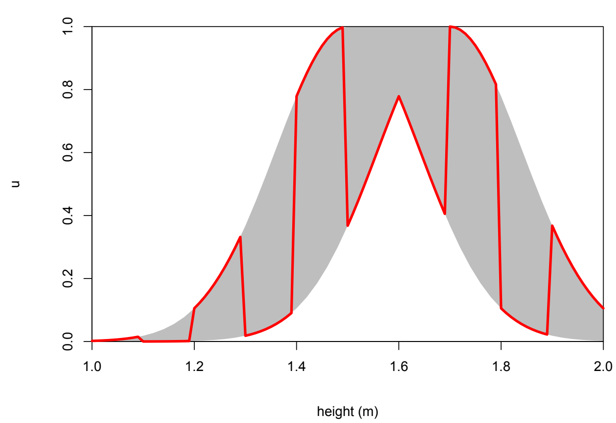

Fig. 8. A discontinuous, non-convex, embedded set within the same FOU as in Figure 7.

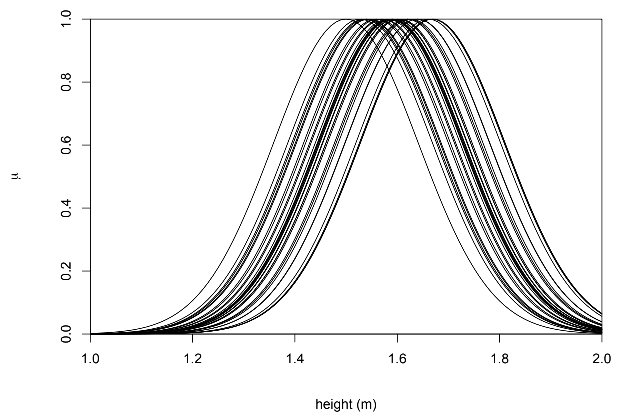

Fig. 9. A possible non-stationary fuzzy set modelling the concept of *medium height* (in metres).

ship functions of the terms within a type-1 fuzzy inference system. For example, in one given system, external semantic or technical requirements may lead to the use of Gaussian forms for all the membership functions of terms, while in another system, perhaps collections of triangular (or piecewise linear) functions may be used. While constraints such as continuity may be imposed on type-2 fuzzy sets, they do not in themselves achieve all the properties we desire (see Section IV-A).

One different form of non-standard fuzzy set which is different from both type-1 and type-2 fuzzy sets is the *non-stationary* fuzzy set, introduced by Garibaldi [7]. A non-stationary fuzzy set consists of a standard type-1 fuzzy set that is perturbed (moves by small amounts) over time. At any instant in time, it is equivalent to a type-1 fuzzy set, but the generating function is non-stationary (there are different forms of non-stationarity, see [7] for more details). Each instant in time at which the non-standard fuzzy set is invoked is termed an *instantiation*. A non-stationary fuzzy set which has been generated with 30 instantiations of the type-1 fuzzy set shown in Figure 1, in which the location (centre of the Gaussian function) has been perturbed left and right over the x-axis by small random amounts, is shown in Figure 9.

A non-stationary fuzzy set overcomes the problem outlined above, in that it is generated by means of perturbing a single *underlying* type-1 membership function, which preserves the semantic meaning of the concept that the generating set is modelling at any given time. It is not a type-2 fuzzy set and thus does not have any equivalent concept corresponding to embedded sets. However, non-stationary fuzzy sets have other problems. The major limitation at present is simply that a non-stationary fuzzy set is *not* a type-2 fuzzy set, although there are conceptual correspondences (see [8]). Consequently, manipulations of non-stationary fuzzy sets require specific mathematical notations and formalisations.

IV. Constrained Type-2 Fuzzy Sets

A. Motivation and Desired Properties

Our motivation for introducing a new form of type-2 fuzzy set can perhaps best be illustrated through the following thought experiment. Suppose we are creating a fuzzy inference system and we wish to model the linguistic variable *height* as part of this system. We determine through knowledge elicitation that *short*, *medium* and *tall* will be the terms of this variable in this context. Furthermore, after a preliminary test, it is determined that Gaussians will be used to model each of the terms (truncated left / right for *short* / *tall*), as these are most appropriate to model these terms. Next, a further knowledge elicitation process takes place in which a number of people are asked for their opinion on the location of the Gaussian for *medium height* by indicating their preferred centre point. One could imagine giving the users a Gaussian outline and asking them to place it in an appropriate place.[1] Now, it seems extremely likely (perhaps even certain) that the people sampled would place the Gaussian in a range of positions. Indeed, one might well see a pattern of superimposed Gaussians such as that shown in Figure 9.

If a conventional type-2 approach utilising interval type-2 fuzzy sets is used to model this scenario, it is not possible to maintain the semantic connection to the type-1 sets that feature in the scenario. If, for example, a general type-2 fuzzy set is represented with an FOU based on the left-most type-1 fuzzy set and the right-most set, and then both continuity and convexity requirements are placed on the set, there is still no relationship to the Gaussian form of the type-1 sets which were the original basis. In order to model such cases, we propose a type-2 fuzzy set with the following properties:

1) A coherency connection with the underlying semantic concept: If there exists an underlying concept (semantic term) that is being modelled through a type-1 fuzzy set of a particular shape (form), for example a Gaussian membership function with specified mean and width, then we require a type-2 set that models the same concept, and hence retains an essential and explicit connection to the original type-1 fuzzy set through sharing the same function form (whatever that may be) in its definition and operations. We tentatively term this property a 'shape coherency'.
2) An embedded representation of variation from a 'mean' of this semantic concept: The type-2 set clearly also models variation in the underlying concept by means of an explicit expression of variation from the original location of the type-1 set. We tentatively term this property 'explicit variability'.

Further properties such as continuity or convexity may be maintained *due to the fact that they are present in the underlying semantic concept expressed through the type-1 set*, but these properties are neither necessary nor sufficient in their own right. Note once again, that we are not claiming that adherence to these properties is in any way *essential* or *necessary* in all situations. Far from it. Rather, we maintain that there are (and will be) situations when such properties are desirable and that, consequentially, there should be a construct available to model these situations.

[1]Of course, one could imagine many alternative methods for performing this elicitation, and we are not stating that this is the only method, or even the best. Obviously, one could vary the width of the Gaussians also, but we do not currently consider this case.

B. Informal Definition

In order to achieve the properties outlined above, we propose a novel form of type-2 fuzzy set which shares features of both a non-stationary fuzzy set and a conventional type-2 fuzzy set. This novel proposal is for a type-2 fuzzy set which is generated through the union of a (finite or infinite) number of continuous *generating* type-1 fuzzy sets, for which the location (position on the x-axis) is altered. This will be termed a *constrained type-2 fuzzy set*, denoted a CT2 fuzzy set, or CT2-FS. Each generating type-1 fuzzy set of a particular constrained type-2 fuzzy set must be of the same form (e.g. the Gaussian function of Figure 1), although different CT2 sets may be generated by different type-1 forms (e.g. triangular, sigmoidal, etc.).

The consequence of this definition is expressed through two constraints. Firstly, the shape of the FOU will be constrained by the possible locations of the generating type-1 fuzzy sets; arbitrary shapes of FOU are not permitted. An FOU of a CT2-FS can only be derived by taking the generating type-1 set and moving it from the left extremum of its allowable positions on the primary domain to the right extremum of its allowable positions on the domain, taking the union of all points between (conceptually, the primary domain may be either continuous or discrete, as for type-1 and type-2 sets). Secondly, the embedded sets are constrained to be exactly the union of the original generating sets, and arbitrary embedded sets are not allowed. So, for example, the FOU of the T2 set shown in Figure 7 is an FOU of an acceptable CT2-FS and the embedded set shown in red is an embedded set of the CT2-FS. Although the FOU shown in Figure 8 is the same as that of Figure 7, the embedded set shown in red in Figure 8 is *not* a valid embedded set of the CT2-FS.

C. Formal Definition

Definition 11: A CT2-FS is a T2-FS denoted by $\tilde{\mu}_\delta : X \times [0,1] \to [0,1]$ that is built assuming that there is a primary membership function and some imprecision about the value of X, that is, from a generator T1-FS $\mu : X \to [0,1]$ (that represents the primary membership function) and a similarity relation $\delta : X \times X \to [0,1]$ (that captures the imprecision related the value of X). And it is built as follows:

$$\tilde{\mu}_\delta(x,u) = \underset{u=\mu(y)}{Sup}\ \delta(x,y)$$ ∎

Remark 3: A key advantage of this definition is that it constrains the possible kinds of T2-FS that can be built to those that can be defined by the generator T1-FS and the similarity relation, and therefore the CT2-FS inherits the properties of them. That is, if both μ and δ are continuous then $\tilde{\mu}_\delta$ is also continuous, if μ and δ are smooth then $\tilde{\mu}_\delta$ is also smooth, and so on. Also, in many applications, an idealised model of the concept (represented by the generator) is known, although it is also known that there is some imprecision in it (captured by the similarity relation). ■

Example 1: Given the concept *medium height* represented by the T1-FS

$$\mu_m(x) = \exp(-(x-1.6)^2/0.2^2)$$

(as shown in Figure 1) and a similarity relation given by

$$\delta(x,y) = \begin{cases} 1; \text{ if } |x-y| \in [0, 0.1] \\ 0; \text{ otherwise} \end{cases}$$

we can built a CT2-FS $\tilde{\mu}_\delta$ as follows:

$$\tilde{\mu}_\delta(x,u) = \underset{u=\mu_m(y)}{Sup} \; \delta(x,y)$$

A graphical representation of this CT2-FS is the interval type-2 set shown in Figure 4. ■

Example 2: Given the same concept *medium height* as in Example 1 above, but with a Gaussian similarity relation given by

$$\delta(x,y) = \begin{cases} \exp(-((x-y)^2)/(0.1/2)^2); \\ \quad \text{if } |x-y| \in [0, 0.1] \\ 0; \text{ otherwise} \end{cases}$$

we can built a CT2-FS $\tilde{\mu}_\delta$ in exactly the same manner:

$$\tilde{\mu}_\delta(x,u) = \underset{u=\mu_m(y)}{Sup} \; \delta(x,y)$$

A graphical representation of this (general) CT2-FS can be seen in Figure 10. ■

Remark 4: A second key advantage of our definition of a constrained type-2 fuzzy set can be seen from these examples in the fact that either the interval type-2 set shown in Figure 4 or the general type-2 set shown in Figure 10 can be generated simply by changing the similarity relation (δ). These two resultant CT2-FS share the same generating function, the same FOU (as per Figure 2) and the same set of embedded type-1 sets: the *only* difference between the two sets are the secondary membership values (and hence the embedded type-2 sets). Furthermore, as they are generated from the same type-1 set as shown in Figure 1, it can also be seen that they are modelling the same underlying concept in two different ways. ■

V. Operations on Constrained Type-2 Fuzzy Sets

A. Negation, Conjunction and Disjunction of CT2-FS

The negation of a CT2-FS can be defined by the negation of the generator function (see Figure 11), that is,

$$\tilde{\mu}'_\delta(x,u) = \tilde{\mu'}_\delta(x,u); \text{ with } \mu' = N(\mu)$$

The conjunction of two CT2-FS can be defined by the conjunction of the generator functions assuming that both share the same imprecision δ (since both share the same primary domain, see Figures 12 and 13), that is,

$$\widetilde{\mu_1}_\delta \wedge \widetilde{\mu_2}_\delta(x,u) = \tilde{\mu}_\delta(x,u); \text{ with } \mu = \mu_1 \wedge \mu_2$$

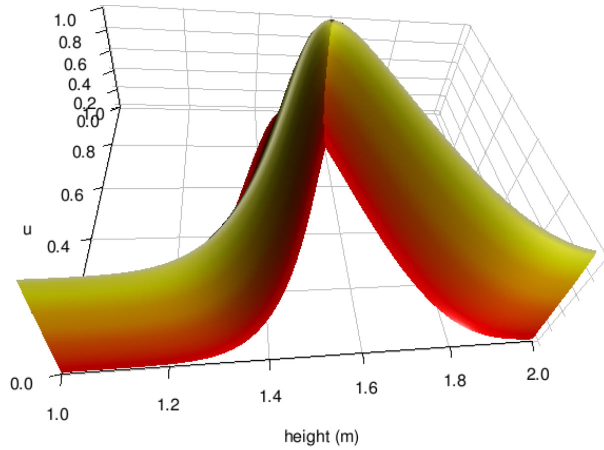

Fig. 10. A CT2-FS generated by moving the generator set with a Gaussian δ function over the range ± 0.10.

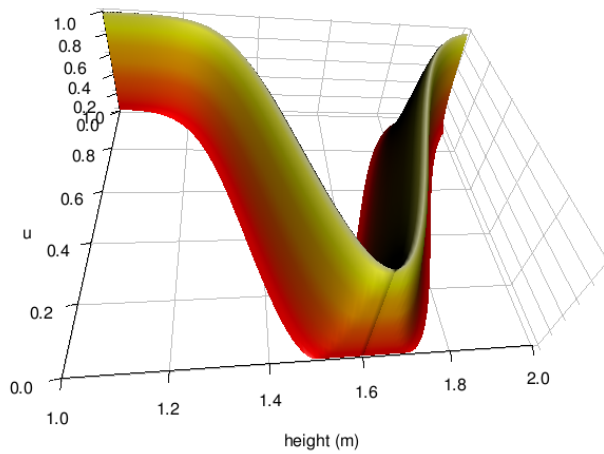

Fig. 11. The negation of the CT2-FS shown in Figure 10.

The disjunction of two CT2-FS can be defined by the disjunction of the generator functions assuming that both share the same imprecision δ (since both share the same primary domain, see Figures 12 and 14), that is,

$$\widetilde{\mu_1}_\delta \vee \widetilde{\mu_2}_\delta(x,u) = \tilde{\mu}_\delta(x,u); \text{ with } \mu = \mu_1 \vee \mu_2$$

B. Inference with CT2-FS

Performing inferences with CT2-FS will require the handling of two different universes of discourse, since rules link the values in antecedent(s) with values at the consequent(s). For example, we can define some rules relating the height of a person with their size of shoes:

If *Height(x)* is *short* then *Shoe-Size(x)* is *small*
If *Height(x)* is *medium* then *Shoe-Size(x)* is *medium*
If *Height(x)* is *tall* then *Shoe-Size(x)* is *large*

In this case we will need to represent all the fuzzy sets involved in the height { *short*, *medium*, *tall* } and for the size of shoes { *small*, *medium*, *large* }. In this case it seems very reasonable to define a similarity relation δ_1 for height and another one δ_2 for the size of shoes, since they are

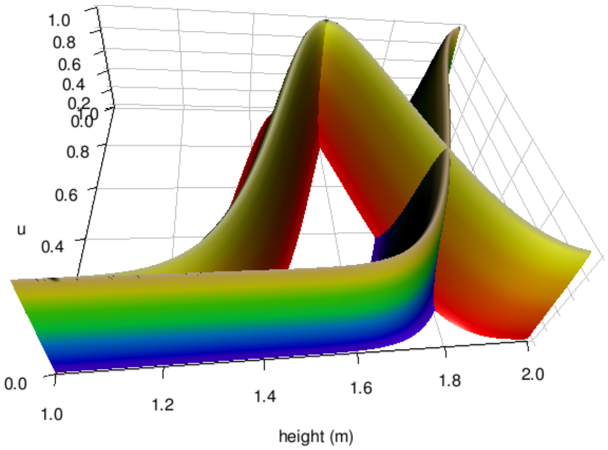

Fig. 12. Two CT2-FS with created from the same generator function (one centred at $x = 1.6$m and one at $x = 2.0$m) and the same δ function.

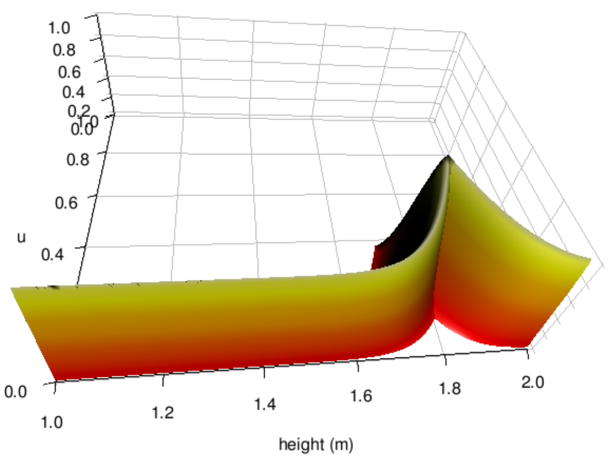

Fig. 13. The conjunction of the two CT2-FS shown in Figure 12.

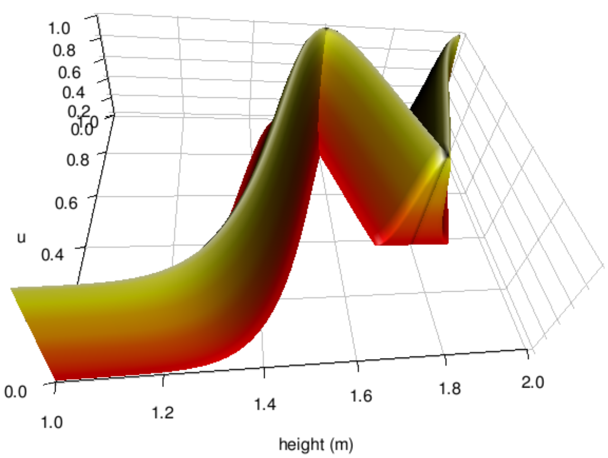

Fig. 14. The disjunction of the two CT2-FS shown in Figure 12.

two different domains and the imprecision involved in each of these does not need to be the same. The inferencing process could then be based on a methodology in which the extension principle is used to extend the inferencing based on the underlying generator functions and the similarity relations. Furthermore, since all similarity relations $\delta(x, y)$ have the value 1 when $x = y$, then (conceptually) it will be possible to perform conventional type-1 inference on the generator sets in the antecedents to produce generators in the consequent(s). We would anticipate that these resultant consequent generators would be present in the type-2 results with secondary memberships of one: these consequential generators could form the basis of constrained type-reduction and defuzzification methods. However, this goes further than the aim of the current paper and is something that needs to be studied in future work.

VI. Discussion

In some real applications, general type-2 fuzzy sets are too general and present some difficulties regarding the semantics of their embedded fuzzy sets. While interval type-2 fuzzy sets constrain T2-FS in some way (by restricting the values of the secondary memberships to either zero or one), they do not overcome these semantic difficulties. This difficulty relates to the fact that in standard type-2 sets $\tilde{\mu}_{\tilde{A}}(x + \delta x, u_{x+\delta x})$ is unrelated to $\tilde{\mu}_{\tilde{A}}(x, u_x)$. Therefore, in this, paper we have constrained general type-2 sets in a novel way to define CT2-FS, in which it is assumed that a idealised model of a generator and the imprecision associated with it are known. These CT2-FS have the advantage of inheriting the properties (such as the continuity) of their building blocks and, furthermore, directly relates a type-2 fuzzy set to an associated type-1 fuzzy set modelling the same underlying concept. As shown in this paper, the same underlying concept of *medium height* can be modelled by a T1-FS, a constrained IT-FS and a constrained T2-FS.

We emphasise that a CT2-FS is a restricted form of a T2-FS, in the same way that a T1-FS or an IT-FS are restricted forms (or special cases) of T2-FS. We believe that CT2-FS may be useful in some contexts, but not all. That is, when there is no natural meaning of the underlying concepts in human reasoning contexts or, for example, in control contexts, then standard interval or general type-2 sets will still be used. Thus, we are not claiming that CT2-FS will in any way replace or invalidate standard type-2 sets or systems, but that they may provide useful and desirable properties with desirable semantics in certain contexts.

The concept of constraining the embedded membership functions resulting from type-2 inference has previously been suggested by Aisbett et al [9]. In this paper, the authors propose a novel method of generating type-2 fuzzy sets based on multivariate modelling, in which "the multivariate modeling underpinning the type-2 fuzzy sets can also constrain realizable forms of membership functions". They go on to demonstrate how constraining the embedded sets produced from conventional type-2 inference results in a tighter subsethood interval, which they claim is "more realistic because it uses only realisable functions". While closely related, the concept of the CT2-FS described in this paper advocates an alternative constraining mechanism based on a different underlying motivation.

We believe the preliminary results presented in Section V demonstrates the potential ease of operating with CT2-FS.

Nevertheless, it is left for future work to study how inference and defuzzification (or type reduction) can be fully defined, and how they can be applied to a real application.

VII. Conclusions and Future Work

A novel form of type-2 fuzzy set, termed the *constrained type-2 fuzzy set*, has been proposed. Further work is required on formalising the definitions of implication, type-reduction and defuzzification to complete the inferencing framework, but this should be relatively straight forward. If the concept is found to be useful, the formal relationships between constrained type-2 fuzzy sets and conventional type-2 fuzzy sets will also have to be established.

Acknowledgments

We thank the participants of the 'Type-2 Fuzzy Logic: State of the Art and Future Directions' workshop, London April 2010, for useful discussions that have helped shape the ideas presented in this paper, and the reviewers for their helpful suggestions.

Sergio Guadarrama's work is been supported by the Spanish Ministry of Education (MEC) under the Human Resources Mobility National Program I-D+i 2008-2011 with reference EX2009-1073.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[2] ——, "The concept of a linguistic variable and its application to approximate reasoning - I," *Information Sciences*, vol. 8, pp. 199–249, 1975.

[3] J. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. NJ: Prentice-Hall, 2001.

[4] S. Guadarrama and D. Pancho, "Learning semantic models from descriptions of shapes using soft constraints," in *IEEE World Congress on Computational Intelligence*, Barcelona, Spain, 2010, pp. 1–8.

[5] A. Pradera, E. Trillas, S. Guadarrama, and E. Renedo, "On fuzzy set theories," in *Fuzzy Logic*, ser. Studies in Fuzziness and Soft Computing, P. Wang, D. Ruan, and E. Kerre, Eds. Springer Berlin / Heidelberg, 2007, vol. 215, pp. 15–47, 10.1007/978-3-540-71258-9_2. [Online]. Available: http://dx.doi.org/10.1007/978-3-540-71258-9_2

[6] N. N. Karnik and J. M. Mendel, "Centroid of a type-2 fuzzy set," *Information Sciences*, vol. 132, pp. 195–200, 2001.

[7] J. M. Garibaldi, M. Jaroszewski, and S. Musikasuwan, "Nonstationary fuzzy sets," *IEEE Transations on Fuzzy Systems*, vol. 16 (4), pp. 1072–1086, 2008. [Online]. Available: http://ima.ac.uk/papers/garibaldi2008a.pdf

[8] ——, "New concepts related to non-stationary fuzzy sets," in *Proceedings of the 2007 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2007)*, 2007, pp. 1684–1689.

[9] J. Aisbett, J. T. Rickard, and D. Morgenthaler, "Multivariate modeling and type-2 fuzzy sets," *Fuzzy Sets and Systems*, vol. 163, pp. 78–95, 2011.