

Generalisations of the Concept of a Non-Stationary Fuzzy Set - a Starting Point to a Formal Discussion

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Abstract—In this paper we propose the concept of an *instantiative fuzzy set* which, in our opinion, constitutes a meaningful addition to the notion of a non-stationary fuzzy set. We begin with a formal definition of an instantiative fuzzy set, and follow specifying basic classes of instantiative fuzzy set operators: the union, the intersection and the complement. Furthermore, we provide formal definitions of selected notions relevant to instantiative fuzzy sets. At the end, we present a subclasses of instantiative fuzzy sets that might be useful for dealing with randomness and vagueness simultaneously. The work presented in this paper is at a very preliminary stage; it is meant as a starting point to a formal discussion on the capture of different facets of uncertainty.

I. INTRODUCTION

IT is well accepted that all humans, including experts, exhibit variation in their decision making. This variation may occur among the decisions of a panel of experts (inter-expert variability) or in the decisions of an individual expert over time (intra-expert variability). For a study of how consistent expert clinicians are in their management of labour, the interested reader is referred to [1].

Type-1 fuzzy sets were introduced by Zadeh, in his seminal paper of 1965 [2], to provide formal tools for dealing with imprecision¹ inherent in real world problems. Although many decision making problems were successfully modelled and implemented using just type-1 fuzzy sets, there were limitations: the ability of type-1 fuzzy sets to capture vagueness is restricted since there is no fuzziness in their membership grades [3].

To address these limitations, Zadeh [4] introduced “fuzzy sets with fuzzy membership functions” - fuzzy sets of type n ($2 \leq n$), for which membership functions range over fuzzy sets of type $n-1$. In practice only type-1 and type-2 fuzzy sets are used, due to excessive computational requirements associated with implementation of higher-type sets. Even type-2 fuzzy sets, although advocated by, for example, Dubois and Prade [5] or Yager [6], had been infrequently employed. This situation has changed only recently, mainly due to the efforts of Mendel [7], but also (possibly) due to the increases in computational power. The interested reader is particularly referred to [7] for a more detailed treatment on type-2 fuzzy sets and systems, and to [8] and [9] for summary tutorials.

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¹In this paper we will use the terms “imprecision” and “vagueness” synonymously.

According to the authors, there are two facets of uncertainty: vagueness and variation. Fuzzy sets were specifically designed to represent vagueness; they do not capture variation in that a fuzzy inference system (FIS) will always produce the same output(s) given the same input(s). This motivated Garibaldi and co-workers to investigate an incorporation of variability into decision making in the context of fuzzy expert systems in medicine [10], [11], [12] and [13]. As a result Garibaldi proposed the notion of *non-deterministic fuzzy reasoning* [14] in which variation is introduced into membership functions of a fuzzy system through random alterations of parameters of these functions. This notion was subsequently formalised through the introduction of *non-stationary fuzzy sets* [15], which were designed to capture variation over time. Initial results indicate that modelling and inferencing capabilities of non-stationary fuzzy sets are fundamentally different from (and complementary to) those provided by type-2 fuzzy sets [15]. For definitions of concepts related to non-stationary fuzzy sets and for a preliminary comparison between non-stationary and type-2 fuzzy sets, the interested reader is referred to [16].

Non-stationary fuzzy sets capture variation over time. In the real world, however, variation is not tied exclusively to time. Moreover, it might pertain to more than one domain. For example, an opinion on a political party will change over the groups in a society and over time. Similarly, a weather forecast will change over data about the current state of the atmosphere, obtained out of sensors measuring precipitation, temperature and wind speed, among others. The capture of variation by non-stationary fuzzy sets is limited to model and implement real-world problems like those. In this paper we introduce instantiative fuzzy sets, an extension of the concept of a non-stationary fuzzy set, to provide a generic tool capable of capturing vagueness and variation.

The layout of the paper is as follows. In Section II definitions of concepts that are relevant to the paper are given (and adapted to the needs of the paper). Specifically, a definition of a type-1 fuzzy set, of a type- n fuzzy set and of a non-stationary fuzzy set is included. Section III is devoted to instantiative fuzzy sets. In that section, a definition of an instantiative type- $n:p$ fuzzy set is provided. The definition is further supported by notes on models of variation and on instantiative fuzzy inference systems. Definitions of basic classes of instantiative fuzzy set operators are given in Section IV. In Section V we establish a vocabulary of terms that might be useful when working with instantiative fuzzy sets. In Section VI a subclass of instantiative fuzzy sets which captures both randomness and vagueness is given. Section

VII concludes the paper.

NOTES ON DENOTATION

We use lower-case letters for the denotation of unidimensional variables and lower-case letters with “-” for the denotation of multidimensional variables. We use upper-case letters to denote sets in general, while an upper-case letter with “-” is used to emphasize that particular set is a set of multidimensional variables. For fuzzy sets, we specify the type of each set in the superscript associated with the letter used to denote the set.

In some places, to avoid ambiguity and confusion, we refer to fuzzy sets introduced by Zadeh as “standard fuzzy sets”. We reserve the term “family” for sets of sets.

II. AUXILIARY DEFINITIONS

Let X denote a universe of discourse. Let $\mathcal{F} \subset 2^{[0,1]^n}$ ($2^{[0,1]^n}$ stands for the powerset of $[0,1]^n$) denote the family of all fuzzy sets of type $n-1$ that have a subset of the $[0,1]$ interval for the universe of discourse.

Definition 1: A type-1 fuzzy set A^1 of the universe of discourse X is characterised by the type-1 membership function $\mu_A^1 : X \rightarrow [0,1]$.

The connection between A_1 and μ_A^1 is denoted as follows:

$$A^1 = \int_{x \in X} \mu_A^1(x)/x.$$

Definition 2: A type- n fuzzy set A^n of the universe of discourse X is characterised by the type- n membership function $\mu_A^n : X \rightarrow \mathcal{F}$.

The connection between A_n and μ_A^n is denoted as follows:

$$A^n = \int_{x \in X} \mu_A^n(x)/x.$$

Let A_1 denote a type-1 fuzzy set of a universe of discourse X characterised by membership function μ_A^1 . We shall express μ_A^1 , without incurring the loss in generality, as $\mu_A^1(x, \bar{\tau})$ where $\bar{\tau} = [\tau_1, \dots, \tau_m]$ denotes the vector of parameters of μ_A^1 and $m \in \mathcal{N}$ denotes the number of parameters. Let T denote a set of time points t_i (possibly infinite) and let $f : T \rightarrow \mathfrak{R}$ denote a perturbation function.

Definition 3: A non-stationary fuzzy set \dot{A} of the universe of discourse X is characterised by the non-stationary membership function $\mu_{\dot{A}} : T \times X \rightarrow [0,1]$ which associates each element (t, x) of $T \times X$ with a t -specific variant of $\mu_A^1(x)$.

An additional restriction is imposed on $\mu_{\dot{A}}$. We require that:

$$\mu_{\dot{A}}(t, x) = \mu_A^1(x, \bar{\tau}(t))$$

where $\bar{\tau}(t) = [\tau_1(t), \dots, \tau_m(t)]$ and $\tau_i(t) = \tau_i + k_i f_i(t)$ for $i = 1, \dots, m$. In this way, each parameter is varied in time by a perturbation function multiplied by a constant.

The connection between \dot{A} and $\mu_{\dot{A}}$ is denoted as follows:

$$\dot{A} = \int_{t \in T} \int_{x \in X} \mu_{\dot{A}}(t, x)/x/t.$$

III. INSTANTIATIVE FUZZY SETS

In this Section the concept we will term an instantiative fuzzy set is introduced. Specifically, a definition of an instantiative type- $n:p$ fuzzy set is given. The definition is accompanied by informal notes on relationships between standard and instantiative, and non-stationary and instantiative fuzzy sets. Furthermore, preliminary remarks pertaining to models of variation and instantiative fuzzy inference systems are included.

A. Instantiative Fuzzy Set Definition

Let A^n denote a type- n ($1 \leq n$) fuzzy set of a universe of discourse X characterised by membership function μ_A^n . We shall express μ_A^n , without incurring the loss in generality, as $\mu_A^n(x, \bar{\tau})$ where $\bar{\tau} = [\tau_1, \dots, \tau_m]$ denotes the vector of parameters of μ_A^n and $m \in \mathcal{N}$ denotes the number of parameters. Let \mathcal{F} stand for the family of all fuzzy sets of type $n-1$ that have a subset of the $[0,1]$ interval for the universe of discourse (if $n = 1$ \mathcal{F} simply denotes the $[0,1]$ interval). Let \mathcal{D} denote a family of sets D_i (domains) where $i = 1, \dots, p$ and $p \in \mathcal{N}$. Let $\bar{D} = D_1 \times \dots \times D_p$ denote the Cartesian product of the elements of \mathcal{D} .

Informally, an instantiative type- $n:p$ fuzzy set is a family of type- n fuzzy sets such that there is a link between the membership functions of the elements of the family.

Definition 4: An instantiative type- $n:p$ fuzzy set $A^{n:p}$ of the universe of discourse X is characterised by the instantiative type- $n:p$ membership function $\mu_A^{n:p} : \bar{D} \times X \rightarrow \mathcal{F}$ which associates each element (\bar{d}, x) of $\bar{D} \times X$ with a \bar{d} -specific variant of $\mu_A^n(x, \bar{\tau})$.

An additional restriction is imposed on $\mu_A^{n:p}$. We require that:

$$\mu_A^{n:p}(\bar{d}, x) = g(\bar{d}, \mu_A^n(x, \bar{\tau}))$$

where $g : D \times \mathcal{F} \rightarrow \mathcal{F}$.

The connection between $A^{n:p}$ and $\mu_A^{n:p}$ shall be denoted as follows:

$$A^{n:p} = \int_{\bar{d} \in \bar{D}} \int_{x \in X} \mu_A^{n:p}(\bar{d}, x)/x/\bar{d}.$$

Alternatively, the second part (after “=”) of the formula denoting the aforementioned connection might be as follows:

$$\int_{d_1 \in D_1} \dots \int_{d_p \in D_p} \int_{x \in X} \mu_A^{n:p}(d_1, \dots, d_p, x)/x/d_p/\dots/d_1.$$

Comments: Any function might be used for g . The only formal limit is that the values of the instantiative membership function $\mu_A^{n:p}$ remain within the \mathcal{F} family. The reader should take note that g does not have to be a single function. For example, g could assume the form of perturbation functions defined on \bar{D} and applied to selected parameters of μ_A^n .

Definition 4 establishes the relationship between standard and instantiative fuzzy sets: for a given type- n fuzzy set A^n and family \mathcal{D} , an instantiative type- $n:p$ fuzzy set $A^{n:p}$ is a set of variants of A^n . These variants are type- n fuzzy sets; each of them corresponds to exactly one element of \bar{D} . We

term a variant of A^n an *instantiation* and denote it by $A_{\bar{d}}^n$ where \bar{d} is the element of \bar{D} corresponding to the variant. The names “instantiation” and “instantiative fuzzy set” have been deliberately selected in order to reflect the relationship between variants of A^n (standard type- n fuzzy sets) and the resulting fuzzy set: for each element $\bar{d} \in \bar{D}$ the corresponding variant is an *instance* of the resulting set. We term fuzzy set A^n - the *underlying fuzzy set* and its membership function μ_A^n - the *underlying membership function*. The link between instantiations is established by the underlying fuzzy set.

From Definition 4 it follows that non-stationary fuzzy sets constitute a subclass of instantiative type-1:1 fuzzy sets. This connection will become more apparent if we make the following assignments between the relevant variables: $n = 1$ (so A^n is a type-1 fuzzy set), $\mathcal{D} = \{T\}$ (thus $p = 1$), $\bar{D} = T$ and $g(\bar{d}, \mu_A^n(x, \bar{\tau})) = \mu_A^1(x, \bar{\tau}(t))$.

B. Models of Variation

There are obviously many ways in which things might change in the real world. Due to the vast array of real-world problems, it is very unlikely that some simple ‘fit them all’, ‘ready to apply’ models of variation exist. In many cases variation can be adequately modelled only after careful analysis and examination of the phenomenon in question, and the resulting model might be (potentially) very complicated. Nevertheless, we believe that the three models of variation proposed below (or a combination thereof) might prove useful in practice (at least for preliminary modelling attempts).

In the formulae below, μ_A^n denotes the membership function of the underlying fuzzy set (A^n) while $\mu_{A_{\bar{d}}}^n$ stands for the membership function of instantiation $A_{\bar{d}}^n$. Symbols $A_{[\alpha+]}^n$ and $A_{\bar{d}[\alpha+]}^n$ denote strong α -cuts² of the underlying fuzzy set A^n and of instantiation $A_{\bar{d}}^n$, respectively. Symbol $v(\bar{d})$ denotes the value of the function $v : \bar{D} \rightarrow \mathfrak{R}$ for element $\bar{d} \in \bar{D}$, while $v_{[\alpha+]}$ denotes the value of the function $v_{[\alpha+] : \bar{D} \rightarrow \mathfrak{R}$ for element $\bar{d} \in \bar{D}$. $\epsilon(\bar{d})$ denotes the value of the function $\epsilon : \bar{D} \rightarrow \mathcal{F}$ for element $\bar{d} \in \bar{D}$.

- Variation in location:

$$\forall \bar{d} \in \bar{D} \{ \mu_{A_{\bar{d}}}^n(x) = \mu_A^n(x + v(\bar{d})) \}.$$

In this model the membership function of the underlying fuzzy set is shifted (as a whole) by $v(\bar{d})$ along the universe of discourse. ■

- Variation in width:

$$\forall \bar{d} \in \bar{D} \{ \forall \alpha \in \mathcal{F} \{ |A_{\bar{d}[\alpha+]}^n| = |A_{[\alpha+]}^n| + v_{[\alpha+]}(\bar{d}) \} \}.$$

In this model the cardinalities of all strong α -cuts of the underlying fuzzy set are adjusted by certain amount. ■

- Noise variation:

$$\forall \bar{d} \in \bar{D} \{ \mu_{A_{\bar{d}}}^n(x) = \mu_A^n(x) \oplus \epsilon(\bar{d}) \}$$

where \oplus denotes a t-conorm operation on instantiative fuzzy sets. ■

²To define (strong) α -cuts for type- n fuzzy sets ($1 < n$), a partial order relation on \mathcal{F} has to be defined.

C. Instantiative Fuzzy Inference Systems

Although this paper is not focussed on a description of inference processes, we make some general points on instantiative fuzzy inference systems. A fuzzy inference system (FIS) consists of the four interconnected components: the fuzzifier, the rules, the inference engine and the defuzzifier (as shown in Fig. 1). Type- n FISs employ type- n fuzzy sets only - FISs which (initially) employ fuzzy sets of different types are of the type of the highest-type fuzzy set in the system, and all other sets are converted to this highest type.

Definition 5: An instantiative fuzzy inference system (iFIS in short) of type $n:p$ is a repetition of standard type- n FISs (Fig. 2). Each constituent FIS employs different instantiations of the underlying membership functions, but all instantiations within such FIS correspond to exactly one and the same element of \bar{D} . ■

Definition 6: A marginal instantiative FIS (miFIS in short) of type $n:p$ is a repetition of standard type- n FISs. Again, each constituent FIS employs different instantiations of the underlying membership functions. All instantiations within a single constituent FIS correspond to exactly one and the same element of a subset $D_i \times \dots \times D_j$ of \bar{D} ($i \in \{1, \dots, p\}$, $j \in \{1, \dots, p\}$ and $i \leq j$). ■

Comments: Implementing an iFIS (miFIS) of type $n:p$ is a matter of iterating over the required number of instantiations while modifying the membership functions of the system. The model of variation and the form of the g function do not have any effect on the inference process.

D. Modelling Example

To illustrate modelling capabilities of instantiative fuzzy sets, let us evoke the following thought experiment.

Let us consider linguistic variable *global surface temperature 2100* that represents a change of the global surface temperature (in K or $^\circ C$) by the year 2100 in comparison to the year 2005. Let us assume that the aforementioned variable is defined on the $[-4, 12]$ interval ($X = [-4, 12]$). Each element $x \in X$ is interpreted as a change of the global surface temperature (in K or $^\circ C$) by the year 2100.

Let us consider two independent experts from different backgrounds. We will denote these backgrounds by 0 and 1 ; they will be represented by set $B = \{0, 1\}$. Let us assume that each expert is asked a question once a year, starting with 2005. The question is: “What will constitute the *average* in the change of the global surface temperature by the year 2100 compared to the year 2005?”. It seems convincing that each experts’s opinion might change on a yearly basis, and that these changes might differ between the experts. Let us assume that the experts have been asked the question three times by now (in the years 2005, 2006 and 2007). The years will be represented by set of time points $T = \{2005, 2006, 2007\}$.

Let us assume that answers take the form of a type-1 fuzzy set with a Gaussian membership function (these functions have two parameters: centre c and standard deviation σ). Let us assume that the underlying membership function $\mu_A^1(x)$

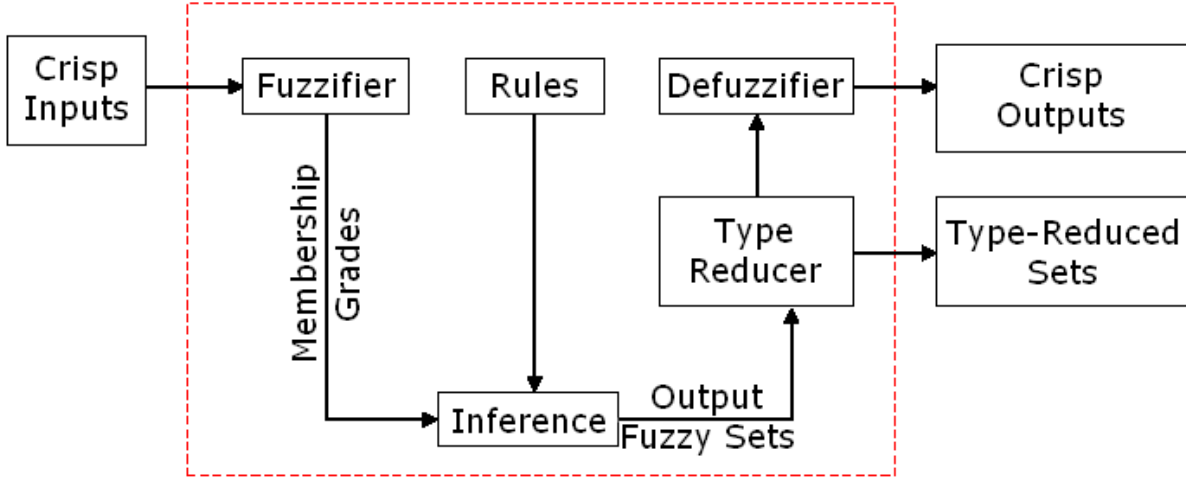


Fig. 1. The mechanisms of a type- n fuzzy inference system (adapted from [7]).

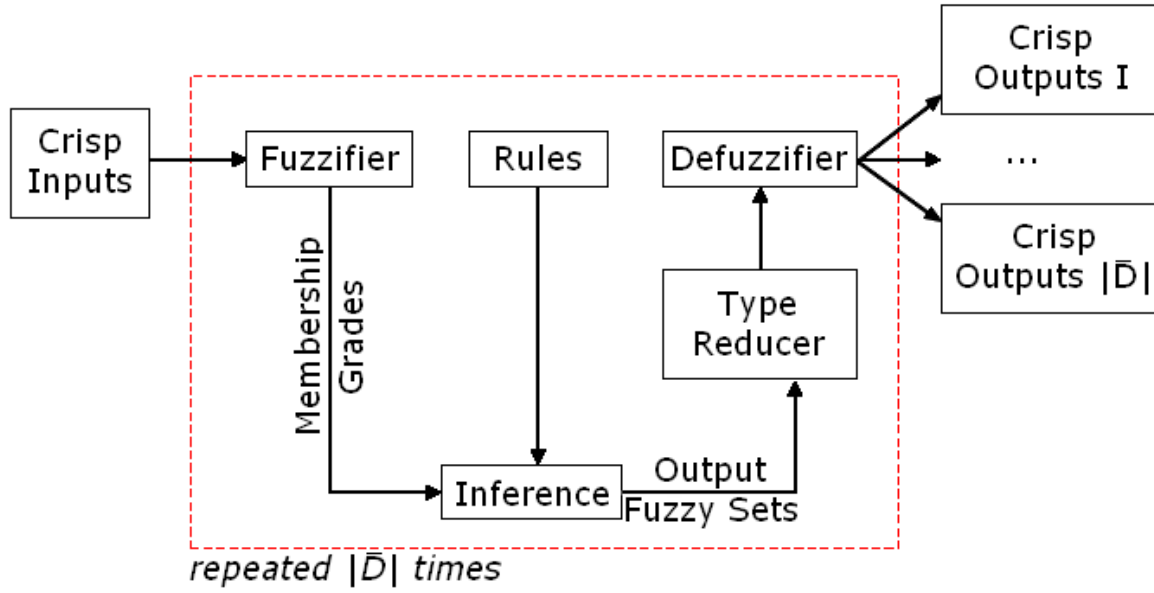


Fig. 2. The proposed mechanisms of an instantiative fuzzy inference system.

is as follows:

$$\mu_A^1(x) = e^{-\frac{(x-4)^2}{2 \cdot 2^2}}.$$

We have that $\mathcal{D} = \{B, T\}$ and $\bar{\mathcal{D}} = B \times T$. There are six instantiations. Each instantiation captures the opinion of one expert in one of the years from T .

An instantiative fuzzy set $A^{1:2}$ representing the opinions of both experts in the years from 2005 until 2007, incorporating variation in the underlying membership function over background and time, might be characterised, for example, by a Gaussian membership function where the centre parameter (c) and standard deviation σ are a function of background

and time:

$$A^{1:2} = \int_{b \in B} \int_{t \in T} \int_{x \in X} e^{-\frac{(x-c(b,t))^2}{2(\sigma(b,t))^2}} / x/t/b.$$

Thus g has the form³ of two perturbation functions: $f_c(b, t)$ and $f_\sigma(b, t)$. Let us assume that $k_c = k_\sigma = 1$, and that $f_c(b, t)$ and $f_\sigma(b, t)$ are defined as follows:

$$f_c(b, t) = \begin{cases} (b + t - 2004) & \text{if } b = 0 \\ -(b + t - 2004) & \text{if } b = 1 \end{cases}$$

³The form of the g function is problem-dependent and can be estimated using problem-specific information (available data).

$$f_\sigma(b, t) = \begin{cases} 1 & \text{if } b = 0 \\ (-b + t - 2004) & \text{if } b = 1 \end{cases}$$

This yields $c(b, t) = 4 + f_c(b, t)$ and $\sigma(b, t) = 2 + \sigma(b, t)$. The resulting instantiative fuzzy set $A^{1:2}$ is depicted in Fig. 3. ■

IV. OPERATIONS ON INSTANTIATIVE FUZZY SETS

In this Section the rules governing use of operators from basic classes of instantiative fuzzy set operators (the union, the intersection and the complement) are described. As an introduction, the familiar rules governing operations on type-1 fuzzy sets are recalled.

Suppose that we have two type-1 fuzzy sets of a universe of discourse X : A^1 and B^1 . These sets are characterised by membership functions $\mu_A^1(x)$ and $\mu_B^1(x)$, respectively:

$$A^1 = \int_{x \in X} \mu_A^1(x)/x,$$

$$B^1 = \int_{x \in X} \mu_B^1(x)/x.$$

The union $A^1 \cup B^1$ of A^1 and B^1 , and the corresponding membership function are as given below:

$$A^1 \cup B^1 = \int_{x \in X} \mu_{A \cup B}^1(x)/x$$

where

$$\forall x \in X \{ \mu_{A \cup B}^1(x) = \mu_A^1(x) \oplus \mu_B^1(x) \}$$

and \oplus denotes a t-conorm.

The intersection $A^1 \cap B^1$ of A^1 and B^1 , and the corresponding membership function are as given below:

$$A^1 \cap B^1 = \int_{x \in X} \mu_{A \cap B}^1(x)/x$$

where

$$\forall x \in X \{ \mu_{A \cap B}^1(x) = \mu_A^1(x) \otimes \mu_B^1(x) \}$$

and \otimes denotes a t-norm.

The complement \bar{A}^1 of A^1 and B^1 , and the corresponding membership function are as given below:

$$\bar{A}^1 = \int_{x \in X} \mu_{\bar{A}}^1(x)/x$$

where

$$\forall x \in X \{ \mu_{\bar{A}}^1(x) = \overline{\mu_A^1(x)} \}$$

and $\bar{}$ denotes a generic complement. ■

Let $\bar{D} = D_1 \times \dots \times D_p$ denote the Cartesian product of the elements of \mathcal{D} , and let $A^{n:p}$ and $B^{n:p}$ denote instantiative type- n : p fuzzy sets of a universe of discourse X :

$$A^{n:p} = \int_{\bar{d} \in \bar{D}} \int_{x \in X} \mu_A^{n:p}(\bar{d}, x)/x/\bar{d}$$

and

$$B^{n:p} = \int_{\bar{d} \in \bar{D}} \int_{x \in X} \mu_B^{n:p}(\bar{d}, x)/x/\bar{d}.$$

Definition 7: The union of $A^{n:p}$ and $B^{n:p}$ is an instantiative type- n : p fuzzy set $A^{n:p} \cup B^{n:p}$ such that:

$$A^{n:p} \cup B^{n:p} = \int_{\bar{d} \in \bar{D}} \int_{x \in X} \mu_{A \cup B}^{n:p}(\bar{d}, x)/x/\bar{d}$$

where

$$\forall_{(\bar{d}, x) \in \bar{D} \times X} \{ \mu_{A \cup B}^{n:p}(\bar{d}, x) = \mu_A^{n:p}(\bar{d}, x) \oplus \mu_B^{n:p}(\bar{d}, x) \}$$

and \oplus denotes a t-conorm. ■

Definition 8: The intersection of sets $A^{n:p}$ and $B^{n:p}$ is an instantiative type- n : p fuzzy set $A^{n:p} \cap B^{n:p}$ such that:

$$A^{n:p} \cap B^{n:p} = \int_{\bar{d} \in \bar{D}} \int_{x \in X} \mu_{A \cap B}^{n:p}(\bar{d}, x)/x/\bar{d}$$

where

$$\forall_{(d, x) \in D \times X} \{ \mu_{A \cap B}^{n:p}(\bar{d}, x) = \mu_A^{n:p}(\bar{d}, x) \otimes \mu_B^{n:p}(\bar{d}, x) \}$$

and \otimes denotes a t-norm. ■

Definition 9: The complement of $A^{n:p}$ is an instantiative type- n : p fuzzy set $\bar{A}^{n:p}$ such that:

$$\bar{A}^{n:p} = \int_{\bar{d} \in \bar{D}} \int_{x \in X} \mu_{\bar{A}}^{n:p}(\bar{d}, x)/x/\bar{d}$$

where

$$\forall_{(\bar{d}, x) \in \bar{D} \times X} \{ \mu_{\bar{A}}^{n:p}(\bar{d}, x) = \overline{\mu_A^{n:p}(\bar{d}, x)} \}$$

and $\bar{}$ denotes a generic complement. ■

The rules specified above are not complete. For example, they do not consider the cases where sets $A^{n:p}$ and $B^{n:p}$ are defined on different universes of discourse X . However, it is impossible to cover all issues related to instantiative fuzzy sets in one paper. This work is meant as an introduction to the subject, and more results will follow.

V. DEFINITIONS OF RELATED CONCEPTS

In this Section we provide formal definitions of selected concepts related to instantiative fuzzy sets. The aim of doing so is to establish a vocabulary of useful terms to be used when working with these sets. Since instantiative fuzzy sets generalise non-stationary fuzzy sets, these concepts are extensions of the concepts introduced in [16]. We will refer some of the new concepts to their non-stationary counterparts and to the concepts relevant to type-2 fuzzy sets [8]. At this point it is important to note that there are many ways to establish a relationship between the concepts relevant to both non-stationary and type-2 fuzzy sets [16]. It follows that there is no simple way to relate the terms important for both inference and type-2 fuzzy sets. Thus any reference to type-2 fuzzy sets should be considered as an example provided for elucidation only.

Definition 10: For each $x \in X$ the set

$$G_x = \bigcup_{\bar{d} \in \bar{D}} \{ (\bar{d}, \mu_A^{n:p}(\bar{d}, x)) \}$$

defines *variational membership function*

$$f_x : \bar{D} \rightarrow \{ \mu_A^{n:p}(\bar{d}, x) \}.$$

We term set G_x the *graph* of f_x . ■

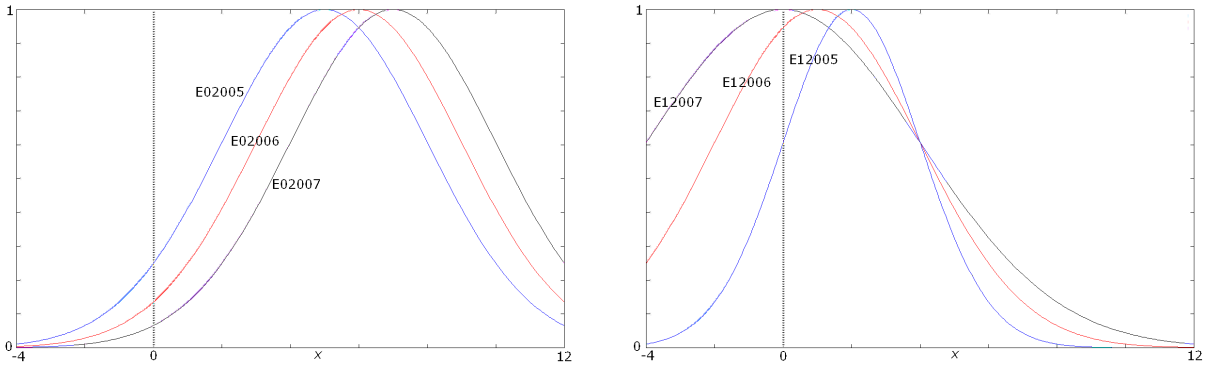


Fig. 3. The instantiative fuzzy set proposed in the example. The figure on the left depicts instantiations representing the answers of the first expert ($b = 0$) while the figure on the right contains instantiations representing the answers of the other expert ($b = 1$).

G_x can be interpreted as a slice of $\bar{D} \times X \times \mathcal{F}$ at point $x \in X$. This concept corresponds to the temporal membership function of non-stationary fuzzy sets, and is reminiscent of the secondary membership function of type-2 fuzzy sets.

Definition 11: For each $x \in X$ the set R_x of the values of f_x :

$$R_x = \bigcup_{\bar{d} \in \bar{D}} \{\mu_A^{n:p}(\bar{d}, x)\}.$$

is termed *variational membership range*. An element of R_x is called *variational membership grade*. ■

The concept of variational membership range corresponds to the temporal membership range of non-stationary fuzzy sets. An element of R_x can be likened to a temporal membership grade (non-stationary sets) or to a secondary grade (type-2 fuzzy sets).

Definition 12: The *lower borderline* $\underline{\mu}_A^{n:p}(x)$ is defined as:

$$\underline{\mu}_A^{n:p}(x) = \bigcup_{x \in X} \{\otimes(R_x)\}$$

where \otimes denotes a t-norm and $\otimes(R_x)$ stands for the result of intersection of all elements of R_x . ■

Definition 13: The *upper borderline* $\overline{\mu}_A^{n:p}(x)$ is defined as:

$$\overline{\mu}_A^{n:p}(x) = \bigcup_{x \in X} \{\oplus(R_x)\}$$

where \oplus denotes a t-conorm and $\oplus(R_x)$ denotes the result of union of all elements of R_x . ■

Definition 14: The *domain of variation* (DOV) is defined as:

$$\text{DOV} = \bigcup_{x \in X} [\underline{\mu}_A^{n:p}(x), \overline{\mu}_A^{n:p}(x)]$$

where $[\underline{\mu}_A^{n:p}(x), \overline{\mu}_A^{n:p}(x)]$ denotes a set of fuzzy sets of type $n-1$ that are *between*⁴ the sets $\underline{\mu}_A^{n:p}(x)$ and $\overline{\mu}_A^{n:p}(x)$. ■

The reader should take note the domain of variation DOV and domain sets D_i are two different concepts.

⁴The domain of variation cannot be defined without a partial order relation on \mathcal{F} .

Definition 15: The *footprint of variation* (FOV) is defined as the set of all values of the instantiative membership function.

$$\text{FOV} = \bigcup_{(\bar{d}, x) \in \bar{D} \times X} \{\mu_A^{n:p}(\bar{d}, x)\} = \bigcup_{x \in X} R_x.$$

■

VI. INSTANTIATIVE FUZZY SETS: SELECTED SUBCLASSES

In addition to non-stationary fuzzy sets, there are other subclasses of instantiative fuzzy sets that might be useful in practice. Below we informally propose one of them.

To this end, let (Ω, S, P) denote a probability space such that $\Omega \subseteq \mathfrak{R}$ (S stands for a σ -algebra over Ω). Let A^n denote a type- n fuzzy set of a universe of discourse X ($1 \leq n$). Let us assume that $\mathcal{D} = \{\Omega\}$ (thus $\bar{D} = \Omega$). An instantiative type- n :1 fuzzy set of the universe of discourse X such that:

$$A^{n:1} = \int_{\omega \in \Omega} \int_{x \in X} \mu_A^{n:1}(\omega, x) / x / \omega$$

captures randomness and vagueness simultaneously.

If we make further assumptions as follows: $n = 1$, g has the form of perturbation functions and $\mu_A^{1:1}(\omega, x) = \mu_A^1(x, \bar{\tau}(\omega))$, we will obtain a concept that is reminiscent of probabilistic fuzzy sets of Hirota [17], and of Meghdadi and Akbarzadeh [18].

Although a thorough comparison of instantiative and probabilistic fuzzy sets is beyond the scope of this paper, let us draw a very coarse line between both kinds of sets.

Hirota [17] defines a probabilistic fuzzy set as a fuzzy set characterised by the membership function which associates each element $(x, \omega) \in X \times \Omega$ with a value from the $[0, 1]$ interval. In addition, for each $x \in X$, $\mu(x)$ is a random variable from Ω to a partially ordered set (most often $[0, 1]$).

There are two differences between type-1:1 instantiative fuzzy sets specified above and Hirota's probabilistic sets:

- 1) Membership functions of type-1:1 instantiative fuzzy sets are defined on the $\Omega \times X$ set, not on the $X \times \Omega$ set.

- 2) There is no formal requirement for the $\mu_A^{1:1}$ membership function to be a random variable (although $\mu_A^{1:1}$ can be defined as such).

Meghdadi and Akbarzadeh [18] define a probabilistic fuzzy set as a fuzzy set characterised by the membership function which is a random variable for each $x \in X$. However, there is no formal requirement that these random variables be defined on the same space of events Ω .

There are two differences between type-1:1 instantiative fuzzy sets defined above, and Meghdadi and Akbarzadeh's sets:

- 1) Membership functions of type-1:1 instantiative fuzzy sets are defined on the $\Omega \times X$ set. In the case of Meghdadi and Akbarzadeh's sets, the membership value for each $x \in X$ is a random variable that might be defined on a different set of events Ω .
- 2) There is no formal requirement for the $\mu_A^{1:1}$ membership function to be a random variable (although $\mu_A^{1:1}$ can be defined as such).

Of course, the subclass of instantiative fuzzy sets proposed in this Section is only an example of how to capture randomness and vagueness at once.

VII. CONCLUSIONS

We introduced non-stationary fuzzy sets [15] as the means of capturing an additional kind of uncertainty compared to standard fuzzy sets; we introduced them to mimic variation inherent in expert decision making [1].

In this paper we have proposed a concept which extends the idea behind non-stationary fuzzy sets - a modelling and reasoning technique able to capture vagueness and variation over multiple domains. We have provided a formal definition of an instantiative fuzzy set. We have adapted the classes of union, intersection and complement operators introduced by Zadeh for type-1 fuzzy sets to instantiative fuzzy sets. We have suggested a set of useful concepts relevant to these sets. Moreover, we have selected two subclasses of instantiative fuzzy sets that might be useful for the simultaneous capture of randomness and vagueness. At the end, we would like to emphasize that the work presented in this paper is at a very preliminary stage and a subject to a scientific discussion.

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REFERENCES

- [1] R. Keith, S. Beckley, J. Garibaldi, J. Westgate, E. Ifeachor, and K. Greene, "A multicentre comparative study of 17 experts and an intelligent computer system for managing labour using the cardiotocogram," *British Journal of Obstetrics and Gynaecology*, no. 102, pp. 688–700, 1995.
- [2] L. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.
- [3] G. Klir and T. Folger, *Fuzzy Sets, Uncertainty and Information*. London: Prentice-Hall, 1988.
- [4] L. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning - I," *Information Sciences*, vol. 8, pp. 199–249, 1975.

- [5] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*. New York: Academic, 1980.
- [6] R. Yager, "Fuzzy subsets of type ii in decisions," *Journal of Cybernetics*, vol. 10, pp. 137–159, 1980.
- [7] J. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. NJ: Prentice-Hall, 2001.
- [8] J. Mendel and R. John, "Type-2 fuzzy sets made simple," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 2, pp. 117–127, 2002.
- [9] J. M. Mendel, R. John, and F. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 6, pp. 808 – 821, December 2006.
- [10] T. Ozen and J. M. Garibaldi, "Investigating adaptation in type-2 fuzzy logic systems applied to umbilical acid-base assessment," in *European Symposium on Intelligent Technologies, Hybrid Systems and Their Implementation on Smart Adaptive Systems*, Oulu, Finland, June 2003.
- [11] T. Ozen, J. M. Garibaldi, and S. Musikasuwana, "Preliminary investigations into modelling the variation in human decision making," in *Information Processing and Management of Uncertainty in Knowledge Based Systems*, Perugia, Italy, July 2004.
- [12] T. Ozen and J. M. Garibaldi, "Effect of type-2 fuzzy membership function shape on modelling variation in human decision making," in *IEEE International Conference on Fuzzy Systems*, Budapest, Hungary, July 2004.
- [13] T. Ozen, J. M. Garibaldi, and S. Musikasuwana, "Modelling the variation in human decision making," in *Fuzzy Sets in the Heart of Canadian Rockies, NAFIPS-04*, Banff, Alberta, Canada, June 2004.
- [14] J. Garibaldi and T. Ozen, "Uncertain fuzzy reasoning: A case study in modelling expert decision making," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 2, pp. 16–30, Feb 2007.
- [15] J. Garibaldi, M. Jaroszewski, and S. Musikasuwana, "Non-stationary fuzzy sets," *submitted to IEEE Transactions on Fuzzy Systems*, 2007.
- [16] —, "New concepts related to non-stationary fuzzy sets," *submitted to IEEE Transactions on Fuzzy Systems*, 2007.
- [17] K. Hirota, "Concepts of probabilistic sets," *Fuzzy Sets and Systems*, vol. 5, pp. 31–46, 1981.
- [18] A. Meghdadi and M. Akbarzadeh-Totonchi, "Probabilistic fuzzy logic and probabilistic fuzzy systems," *Proceedings of the 2001 IEEE International Fuzzy Systems Conference*, pp. 1127–1130.