Nonstationary Fuzzy Sets

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***Abstract*—In this paper, the notion termed a "nonstationary fuzzy set" is introduced, and the concept of a perturbation function that is used for generating nonstationary fuzzy sets is presented. Definitions of the basic set operators (the union, the intersection, and the complement) for nonstationary fuzzy sets are given, together with proofs of selected properties of these operators. Two case studies were carried out in order to illustrate the use of nonstationary fuzzy sets in a nonstationary fuzzy inference, and to provide an initial insight into the relationships between nonstationary and interval type-2 fuzzy sets.**

***Index Terms*—Nonstationary fuzzy sets, perturbation functions, type-2 fuzzy sets.**

I. Introduction

Fuzzy sets were first introduced by Zadeh [1] to model the uncertainty inherent in assigning memberships of elements to real-world sets such as the set of *old* people or the set of *tall* people. They were specifically designed to represent vagueness, and provided formalized tools for dealing with imprecision in real-world problems. In practice, however, although many complex decision-making problems can be expressed by using just type-1 fuzzy sets, there remain limitations. The ability of type-1 fuzzy sets to model and/or minimize the effects of uncertainty is restricted, as there is actually no *fuzziness* in the standard type-1 membership grade. This issue has been pointed out by many people, including Klir and Folger [2], who stated:

> "... it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers."

Zadeh addressed this problem originally in his seminal papers of 1975, in which he introduced the concept of linguistic variables [3]. Zadeh proposed "fuzzy sets with fuzzy membership functions," and went on to define fuzzy sets of type n, $n = 2, 3, \ldots$, for which membership functions range over fuzzy sets of type $n - 1$. The use of type-2 fuzzy sets was subsequently advocated by Dubois and Prade [4]:

> "To take into account the imprecision of membership functions, we may think of using a type-2 fuzzy set"

and Yager [5]:

> "The usefulness of fuzzy subsets of type II [type-2] is that it enables us to extend membership grades to linguistic values."

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Since their introduction, theoretical aspects relevant to type-2 fuzzy sets have been investigated by Dubois and Prade [4], and Mizumoto and Tanaka [6], [7].

However, their use in practice has been limited due to the significant computational requirements associated with their implementation. More recently, type-2 sets have received renewed interest, mainly due to the efforts of Mendel [8], but also, possibly, due to the increase in computational power. Mendel has established a set of terms to be used when working with type-2 fuzzy sets, and in particular, introduced a concept known as the *footprint of uncertainty* (FOU) that provides a useful verbal and graphical description of the uncertainty captured by any given type-2 set. The interested reader is particularly referred to [8] for a more detailed treatment, and to [9] and [10] for summary tutorials. Mendel has particularly concentrated on a restricted class of type-2 fuzzy sets known as *interval* type-2 fuzzy sets. Interval type-2 fuzzy sets are characterized by secondary membership functions that only take the value of 1 over their domain (the primary membership of x). This restriction greatly reduces the computational costs of performing inference compared with general type-2 sets. Moreover, Mendel has provided closed-form formulas for the union, the intersection, and the complement of interval type-2 fuzzy sets, and computational algorithms for type reduction (necessary for type-2 defuzzification).

While type-2 fuzzy sets capture the concept of uncertainty in membership functions by introducing a range of membership values associated with each value of the base variable, they do not capture the notion of variability, in that a type-2 fuzzy inference system (FIS) will always produce the same output(s), given the same input(s) (although any outputs will, of course, be type-2 fuzzy sets with an implicit representation of uncertainty). On the other hand, it is well accepted that all humans, including "experts," exhibit variation in their decision making. Variation may occur among the decisions of a panel of human experts (interexpert variability), as well as in the decisions of an individual expert over time (intraexpert variability). Up until now, there has been an implicit assumption that expert systems, including fuzzy expert systems, should not exhibit such a variation. This motivated Garibaldi *et al.* to investigate the incorporation of variability into decision making in the context of fuzzy expert systems in a medical domain [11]–[15]. In these papers, the informal notion of *nondeterministic fuzzy reasoning*, in which the variability is introduced into the membership functions of a fuzzy system through random alterations to the parameters of these functions, was proposed. We now formalize and extend this research through the introduction of a notion that we term a *nonstationary fuzzy set*.

II. Type-1 and Type-2 Fuzzy Sets

Type-1 fuzzy sets are characterized by membership functions of one argument (they can be represented graphically in two

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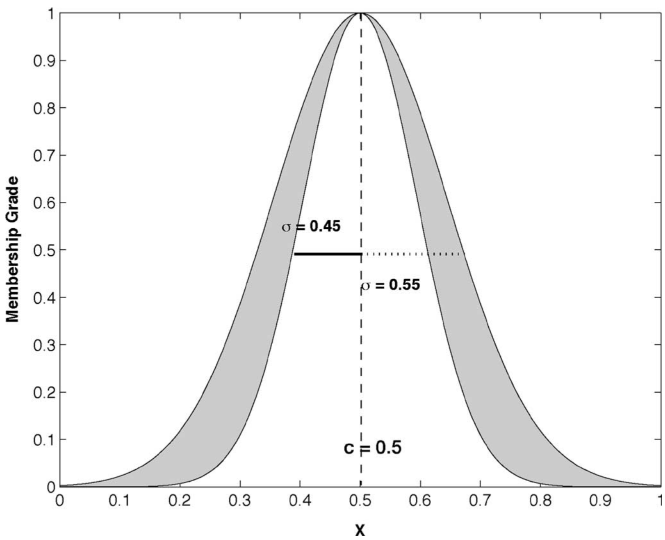

Fig. 1. Illustration of the FOU of a Gaussian interval type-2 fuzzy set generated by varying the standard deviation of the Gaussian.

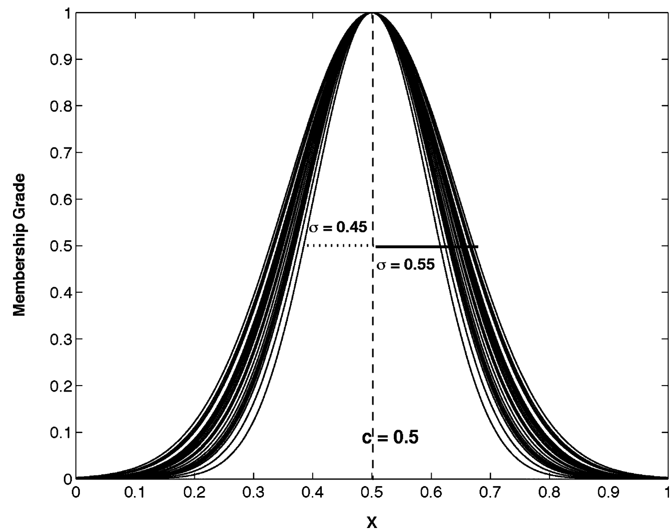

Fig. 2. Illustration of a Gaussian nonstationary fuzzy set featuring variation in standard deviation and instantiated 30 times.

dimensions), which map each element of the set to the membership grade that is a crisp number in $[0, 1]$. Type-2 fuzzy sets are characterized by membership functions of two arguments (informally, they can be said to introduce a third dimension). The third dimension provides an additional degree of freedom, which allows the representation of different (additional) information; the membership grade of each element of a type-2 fuzzy set is a fuzzy set in [0,1].

The uncertainty in primary memberships of an interval type-2 fuzzy set can be represented as a bounded region, termed the FOU [16]. An example of the FOU is shown in Fig. 1 (the shaded, grey area). The lower bound of the FOU has been generated by the Gaussian with the center of 0.5 and the standard deviation of 0.45, and the upper bound of the FOU has been generated by the Gaussian with the same center and the standard deviation of 0.55. Type-2 fuzzy sets are useful in circumstances where it is difficult to determine the exact membership function for a fuzzy set. For a specific example, the reader is referred to [17].

III. Nonstationary Fuzzy Sets and Systems

As mentioned in Section I, Garibaldi proposed the notion of "nondeterministic fuzzy reasoning," in which the variability is introduced into the membership functions of a fuzzy system through random alterations to the parameters of these functions. In this section, the aforementioned notion is extended and formalized through the introduction of a concept that we will term a *nonstationary fuzzy set*. Informally, a nonstationary fuzzy set is a set (collection) of type-1 fuzzy sets in which there is a connection between (or restriction on) the membership functions of the fuzzy sets. This connection is expressed as a slight variation in the membership function over time. Fig. 2 shows a pictorial representation of repeated instantiations of a nonstationary fuzzy set in which a Gaussian membership function has variation in its standard deviation. The sets were obtained by repeatedly generating (30 times) a Gaussian membership function with the center of 0.5 and the standard deviation that varies between 0.45 and 0.55. That is, the parameters of the nonstationary set have been chosen in this example so that the extreme parameter values would match those used to generate the lower and the upper bounds shown in Fig. 1. It is apparent from Figs. 1 and 2 that the union of all possible instantiations of the nonstationary fuzzy set is reminiscent of the FOU of the type-2 set. However, it is important to emphasize that nonstationary fuzzy sets are *not* type-2 fuzzy sets. Essentially, type-2 fuzzy sets are "fuzzy sets with fuzzy membership functions" [3, p. 241], while nonstationary fuzzy sets are collections of related fuzzy sets. From a formal point of view, nonstationary fuzzy sets are defined in a different way than type-2 fuzzy sets, and have distinct properties (as will be discussed in Section III-A and III-B, and particularty as illustrated in Example 2, Section III-D). From a modeling point of view, they model different things: nonstationary fuzzy sets model *temporal variability* in (type-1) membership functions, while type-2 fuzzy sets model uncertain membership functions.

A. Nonstationary Fuzzy Sets

Let A denote a fuzzy set of a universe of discourse X characterized by a membership function μ_A. Let T be a set of time points t_i (possibly infinite) and $f : T \to \Re$ denote a *perturbation function.*

Definition 1: A *nonstationary fuzzy set* $\dot{A}$ of the universe of discourse X is characterized by a *nonstationary membership function* $\mu_{\dot{A}} : T \times X \to [0, 1]$ that associates with each element (t, x) of $T \times X$ a time-specific variation of $\mu_A(x)$. The nonstationary fuzzy set $\dot{A}$ is denoted by

$$\dot{A} = \int_{t \in T} \int_{x \in X} \mu_{\dot{A}}(t, x)/x/t.$$

However, an additional restriction is imposed on $\mu_{\dot{A}}$. To formulate it in a coherent and precise way, let us first notice that $\mu_A(x)$ can be expressed as $\mu_A(x, p_1, \dots, p_m)$, where $p_1, \dots, p_m$ denote the parameters of $\mu_A(x)$. Now, we require that

$$\mu_{\dot{A}}(t, x) = \mu_A(x, p_1(t), \dots, p_m(t))$$

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where $p_i(t) = p_i + k_i f_i(t)$ and $i = 1, \ldots, m$. In this way, each parameter is varied in time by a perturbation function multiplied by a constant. ■

This definition establishes a relationship between standard and nonstationary fuzzy sets. Specifically, for a given standard fuzzy set A and a set of time points T, a nonstationary fuzzy set $\dot{A}$ is a set of duplicates of A varied over time. We term a time duplicate of A an *instantiation* and denote it by $\dot{A}_t$, so that $\dot{A}_t(x) = \dot{A}(t, x)$. Thus, at any given moment of time $t \in T$, the nonstationary fuzzy set $\dot{A}$ instantiates the standard fuzzy set $\dot{A}_t$. We term the standard fuzzy set A the *underlying* fuzzy set, and its associated membership function $\mu_A(x)$ the *underlying* membership function.

B. Perturbation Functions

The original intention behind nonstationary fuzzy sets was to capture minor variations of a membership function corresponding to subtle differences in opinion over time. Additionally, the intention was that a nonstationary fuzzy set remains close to the underlying fuzzy set over time; that is, there is no permanent "drift" or alteration of the membership function which is characteristic of learning processes.[1] Thus, the term "perturbation function" was deliberately chosen to imply that parameter changes induced by the function are "small" or, more precisely, that parameter changes induce "small" and temporary alterations in $\mu_A(x)$.

There are many ways in which an opinion may vary over time. Three alternative forms of the nonstationarity that might be useful in practice can be formalized as follows:

1) variation in location

$$\forall_{t\in T}\ \mu_{\dot{A}}(t, x) = \mu_A(x + a(t))$$

where $a(t)$ is a constant for any given t. Thus, the membership function is shifted, as a whole, left $(a(t) > 0)$ or right $(a(t) < 0)$ by small amounts along the universe of discourse, relative to the underlying membership function.

2) variation in width

$$\forall_{t\in T}, \forall_{\alpha\in[0,1]}\ |\dot{A}_{t,\alpha+}| = |A_{\alpha+}| + a_\alpha(t)$$

where $a_\alpha(t)$ is a constant for any given α and t. In this case, the cardinalities of all strong α-cuts are increased $(a_\alpha(t) > 0)$ or decreased $(a_\alpha(t) < 0)$ by small amounts, relative to the underlying membership function.

3) noise variation

$$\forall_{t\in T}\ \mu_{\dot{A}}(t, x) = \mu_A(x) + a(t).$$

where $a(t)$ is a constant for any given t. Thus, the membership function is shifted upward $(a(t) > 0)$ or downward $(a(t) > 0)$, relative to the underlying membership function.

The next issue to be addressed is the form of the perturbation function. In general, it would appear that any function of time might be used as a perturbation function within the formal restriction that the membership function remains in bounds $(\mu_A(t, x) \in [0, 1])$. In theory, even a true random function could be a perturbation function. Given that any measurement of time is arbitrary and relative, the actual set of functions that might be useful in practice is more restrictive. A few families of perturbation functions that might be useful in practice are (with examples):

1) periodic:

$$f(t) = \sin(\omega t + \alpha); \tag{1}$$

2) pseudorandom:

$$f(t) = \frac{s(t+1) - 2^{47}}{2^{47}} \tag{2}$$

where $s(0)$ is the initial seed in $[0, 2^{48}]$ and

$$s(t+1) = (25,214,903,917s(t) + 11) \mod 2^{48};$$

3) differential time-series such as the Mackey–Glass equation given by

$$\frac{df(t)}{dt} = \frac{0.2f(t-\tau)}{1 + f^{10}(t-\tau)} - 0.1f(t)$$

where τ is a constant.

Any units might be used for time t, but the most natural would be to express time in seconds, in the absence of any good reason not to. Again, given that any physical notion of time is relative, any arbitrary point in time might be chosen as zero.

C. Nonstationary Fuzzy Inference Systems

Although this paper is not focused on a complete description of the fuzzy inference process, we make some general points in order to clarify the difference between type-1, type-2, and nonstationary fuzzy sets. An FIS consists of four main interconnected components: the *rules*, the *fuzzifier*, the *inference engine*, and an *output processor*. Type-1 FISs use only type-1 fuzzy sets, whereas an FIS that uses at least one type-2 fuzzy set is called a type-2 FIS. Fig. 3 shows the mechanisms of a type-2 FIS (adapted from [8]). Fig. 4 shows the mechanisms of the inference process in an FIS consisting of nonstationary fuzzy sets. We naturally term such an FIS a "nonstationary FIS." It should be emphasized that a nonstationary FIS is simply a repetition of a type-1 FIS with slightly different instantiations of the membership functions over time. Thus, implementing a nonstationary FIS is simply a matter of iterating over the required number of instantiations while perturbing the membership functions. Neither the form of nonstationarity (variation in location, variation in width, or noise variation) nor the form of perturbation function (periodic, random, chaotic, etc.) has any effect on the inference process. Hence, an inference with nonstationary fuzzy sets is clearly different from the type-2 inference, and does not suffer the difficulties of type-2 inference (particularly the inference using general type-2 fuzzy sets).

[1] Note that there is an interesting relationship between small variations over time that are proposed here and long-term changes in membership functions that are seen in adaptive (or "learning") fuzzy sets. Such relationships are outside the scope of the present paper and might require further research.

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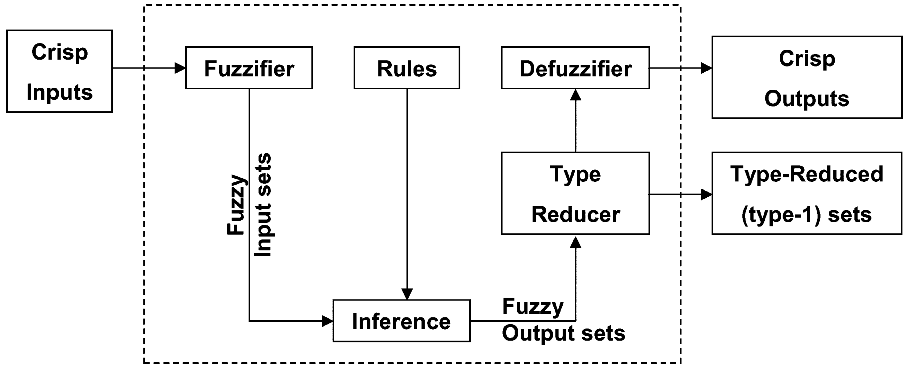

Fig. 3. Mechanisms of a type-2 FIS (adapted from [8]).

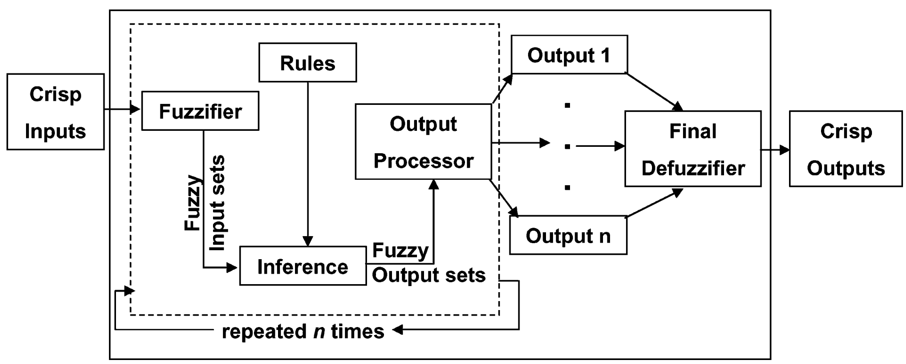

Fig. 4. Proposed mechanisms of a nonstationary FIS.

D. Examples

Example 1: As an example, let us illustrate the three forms of nonstationarity in the context of Gaussian membership functions. A standard Gaussian membership function can be written in the parameterized notation as

$$\mu(x, c, \sigma, \epsilon) = e^{-(x-c)^2/2\sigma^2} + \epsilon.$$

(Of course, normally ϵ is zero.) Now, the three forms of nonstationarity described earlier can be, respectively, expressed by

$$\mu(x, c(t), \sigma, \epsilon) = e^{-(x-c(t))^2/2\sigma^2} + \epsilon$$
$$\mu(x, c, \sigma(t), \epsilon) = e^{-(x-c)^2/2\sigma(t)^2} + \epsilon$$
$$\mu(x, c, \sigma, \epsilon(t)) = e^{-(x-c)^2/2\sigma^2} + \epsilon(t).$$

Note that, for simplicity, we omit (t) from any parameter that does not vary over time. Naturally, there is no reason why these three different kinds of variation could not be combined together as

$$\mu(x, c(t), \sigma(t), \epsilon(t)) = e^{-(x-c(t))^2/\sigma(t)^2} + \epsilon(t)$$

but, for simplicity, we do not consider such combined nonstationarity at present. ■

Example 2: To illustrate the differences in modeling between nonstationary and interval type-2 fuzzy sets, consider the following thought experiment. Let us consider the linguistic variable *middle age*, representing the opinion of an individual on what constitutes middle age. Let us assume that the individual is asked for an opinion, in the form of the center c and standard deviation σ of a Gaussian membership function once a week. It seems plausible that the individual's opinion might change on a weekly basis. Let us also assume that the universe of discourse X is the interval $[0, 100]$, with x interpreted as the age (in years). Moreover, let us assume that the individual gives the answers as in Table I.

TABLE I
ANSWERS GIVEN BY THE INDIVIDUAL OVER THE COURSE OF FIVE WEEKS

Week	M.F.	c	σ
1	G_1	50	4
2	G_2	48	4
3	G_3	50	5
4	G_4	52	4
5	G_5	50	6

The nonstationary fuzzy set $\dot{A}$, representing the opinion of the individual over the five weeks, incorporating variability in the underlying membership function, might be represented by a Gaussian membership function where the center and the standard deviation parameters are functions of time as

$$\dot{A} = \int_{t \in T} \int_{x \in X} e^{-\frac{(x-c(t))^2}{2\sigma(t)^2}} / x / t.$$

There are five instantiations corresponding to the five weeks: $\dot{A}_1 = G_1$, $\dot{A}_2 = G_2$, $\dot{A}_3 = G_3$, $\dot{A}_4 = G_4$, and $\dot{A}_5 = G_5$, and $T = \{1, 2, 3, 4, 5\}$. Then, the single underlying membership

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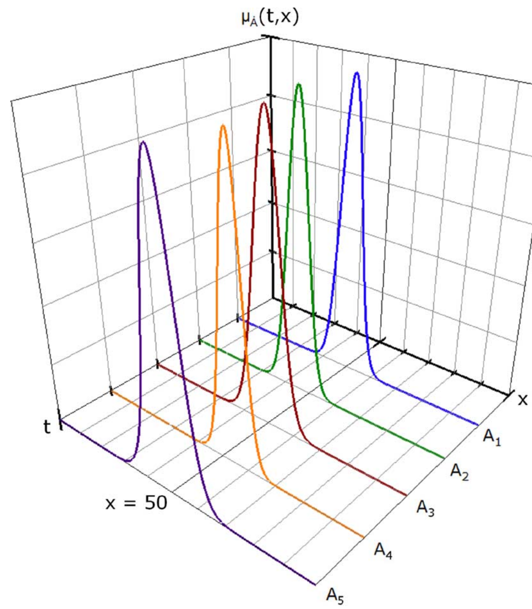

Fig. 5. Nonstationary fuzzy set proposed in Example 2.

function $\mu_A(x)$ could be

$$\mu_A(x) = e^{-(x-50)^2/2(5^2)}$$

with the perturbation function $f_c(t)$ defined as

$$f_c(t) = \begin{cases} 0, & \text{if } t \in \{1,3,5\} \\ -2, & \text{if } t = 2 \\ 2, & \text{if } t = 4 \end{cases}$$

which, with $k_c = 1$, yields $c(t) = 50 + f_c(t)$. Further, the perturbation function $f_\sigma(t)$ would then be defined as

$$f_\sigma(t) = \begin{cases} -1, & \text{if } t \in \{1,2,4\} \\ 0, & \text{if } t = 3 \\ 1, & \text{if } t = 5 \end{cases}$$

which, with $k_\sigma = 1$, yields $\sigma(t) = 5 + f_\sigma(t)$. The resulting nonstationary fuzzy set $\dot{A}$ is depicted in Fig. 5.

Now, consider how this might be modeled using type-2 fuzzy sets. The type-2 fuzzy set $\tilde{A}$ might feature primary memberships J_x given by

$$J_x = \bigcup_{i=1}^{5} \{G_i(x)\} \quad \forall x \in X.$$

The next question to address is the form of the secondary membership function. Given that there is no apparent weighting between the different opinions, one might suggest using an interval type-2 set in which all secondary membership values are 1. However, two points emerge. First, we have to introduce secondary memberships for no apparent reason. Second, since the J_xs in type-2 fuzzy sets are independent, the information on the shape of the original membership functions has been lost. ■

We believe that the nonstationary fuzzy sets proposed here provide a simpler mechanism for modeling such a variation in opinion. As such, we believe that they represent an *additional* modeling technique that is fundamentally different from (and complementary to) type-2 fuzzy sets.

IV. Operations on Nonstationary Fuzzy Sets

In this section, the operators of *union*, *intersection*, and *complement* of nonstationary fuzzy sets are introduced. To this end, we first recall the familiar properties of type-1 fuzzy sets. Suppose that we have two fuzzy sets, A and B, characterized by membership functions $\mu_A(x)$ and $\mu_B(x)$ as

$$A = \int_{x\in X} \mu_A(x)/x$$

$$B = \int_{x\in X} \mu_B(x)/x.$$

Recall that

$$A \cup B = \int_{x\in X} \mu_{A\cup B}(x)/x$$

$$A \cap B = \int_{x\in X} \mu_{A\cap B}(x)/x$$

$$\bar{A} = \int_{x\in X} 1 - \mu_A(x)/x.$$

The membership functions of the union and intersection of A and B and the complement of A are, of course,

$$\mu_{A\cup B}(x) = \mu_A(x) \oplus \mu_B(x) \quad \forall x \in X$$

where $\oplus$ is a t-conorm,

$$\mu_{A\cap B}(x) = \mu_A(x) \otimes \mu_B(x) \quad \forall x \in X$$

where $\otimes$ is a t-norm, and

$$\mu_{\bar{A}}(x) = \overline{\mu_A(x)} \quad \forall x \in X$$

where $^-$ denotes a generic complement. Using the maximum t-conorm, the minimum t-norm, and the standard complement, the previous equations become

$$\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)] \quad \forall x \in X$$

$$\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)] \quad \forall x \in X$$

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x) \quad \forall x \in X.$$

Now, let $T = \{t_1, \ldots, t_n\}$ be a set of time points t_i, and let $\dot{A}$ and $\dot{B}$ be nonstationary fuzzy sets of a universe of discourse X. Thus

$$\dot{A} = \int_{t\in T}\int_{x\in X} \mu_{\dot{A}}(t,x)/x/t$$

and

$$\dot{B} = \int_{t\in T}\int_{x\in X} \mu_{\dot{B}}(t,x)/x/t.$$

Definition 2: The union of $\dot{A}$ and $\dot{B}$ is a nonstationary fuzzy set $\dot{A} \cup \dot{B}$ such that

$$\dot{A} \cup \dot{B} = \int_{t\in T}\int_{x\in X} \mu_{\dot{A}\cup\dot{B}}(t,x)/x/t$$

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TABLE II
XOR TRUTH TABLE

	Input1	*Input2*	XOR (*Input1*, *Input2*)
Combination I	0	0	0
Combination II	0	1	1
Combination III	1	0	1
Combination IV	1	1	0

where

$$\mu_{\dot{A}\cup\dot{B}}(t,x) = \mu_{\dot{A}}(t,x) \oplus \mu_{\dot{B}}(t,x)\ \forall(t,x) \in T \times X.$$

Using the maximum t-conorm, this becomes

$$\mu_{\dot{A}\cup\dot{B}}(t,x) = \max[\mu_{\dot{A}}(t,x), \mu_{\dot{B}}(t,x)]\ \forall(t,x) \in T \times X.$$

Definition 3: The intersection of $\dot{A}$ and $\dot{B}$ is a nonstationary fuzzy set $\dot{A} \cap \dot{B}$ such that

$$\dot{A} \cap \dot{B} = \int_{t\in T} \int_{x\in X} \mu_{\dot{A}\cap\dot{B}}(t,x)/x/t$$

where

$$\mu_{\dot{A}\cap\dot{B}}(t,x) = \mu_{\dot{A}}(t,x) \otimes \mu_{\dot{B}}(t,x)\ \forall(t,x) \in T \times X$$

which, using the minimum t-norm, becomes

$$\mu_{\dot{A}\cap\dot{B}}(t,x) = \min[\mu_{\dot{A}}(t,x), \mu_{\dot{B}}(t,x)]\ \forall(t,x) \in T \times X.$$

Definition 4: The complement of $\dot{A}$ is a nonstationary fuzzy set $\bar{\dot{A}}$ such that

$$\bar{\dot{A}} = \int_{t\in T} \int_{x\in X} \mu_{\bar{\dot{A}}}(t,x)/x/t$$

where

$$\mu_{\bar{\dot{A}}}(t,x) = \overline{\mu_{\dot{A}}(t,x)}\ \forall(t,x) \in T \times X$$

which, using the standard complement, becomes

$$\mu_{\bar{\dot{A}}}(t,x) = 1 - \mu_{\dot{A}}(t,x)\ \forall(t,x) \in T \times X.$$

V. CASE STUDIES OF NONSTATIONARY INFERENCE

In this section, two case studies are described that were carried out to demonstrate some properties of the nonstationary fuzzy inference, and to illustrate the differences in inference processes between nonstationary and interval type-2 fuzzy sets. In both studies, all FISs were constructed to predict the truth value of the fuzzy equivalent of the classical XOR operation (Table II), where two input and one output variables could take any value in the $[0, 1]$ range. Each FIS consisted of two input variables (*Input1* and *Input2*), one output variable (*Output*) and the following four rules.

1) If *Input1* is *low* and *Input2* is *low* then *Output* is *low*.
2) If *Input1* is *low* and *Input2* is *high* then *Output* is *high*.
3) If *Input1* is *high* and *Input2* is *low* then *Output* is *high*.
4) If *Input1* is *high* and *Input2* is *high* then *Output* is *low*.

TABLE III
INPUT VALUES FOR FUZZY SYSTEMS

	Input1	*Input2*
Combination I	0.25	0.25
Combination II	0.25	0.75
Combination III	0.75	0.25
Combination IV	0.75	0.75

Each variable consisted of two membership functions corresponding to the meanings of the terms *low* and *high*. The four combinations of input values used throughout the case studies are shown in Table III.

A. Notation

The nonstationary FISs are denoted by NS1-#$-&*. "#" is either "$G$" to denote a Gaussian underlying membership function or "T" for a triangular one, "$" is either "$L$" to denote variation in location or "W" for variation in width, "&" is either "U" to denote a uniform perturbation function, "G" for a Gaussian or "S" for a sinusoidal perturbation function, and finally, "*" denotes the number of instantiations. For example, NSI-GL-G100 denotes a nonstationary FIS that utilizes a Gaussian underlying membership function, features variation in location, and a Gaussian perturbation function, instantiated 100 times.

The interval type-2 systems are denoted by IT2-#$. "#" is either "G" to denote the FOUs derived from Gaussians or "T" for triangular ones, and "$" is either "$L$" to denote variation in location or "W" for variation in width.

B. Case Study I: Gaussian Membership Functions

1) Nonstationary FISs: In the first case study, nonstationary FISs utilizing Gaussians of the form

$$\mu_A(x, c, \sigma) = e^{-(x-c)^2/2\sigma^2} \tag{3}$$

as the underlying membership functions were investigated (note that the ϵ parameter has now been dropped as the noise variation was not considered in this study).

The *low* underlying membership function had center 0.3, the *high* underlying membership function had center 0.7, and both had a standard deviation of 0.1. These functions are shown in Fig. 6. Two forms of nonstationarity were implemented as follows:

1) variation in location: only the center c of the Gaussian (3) was varied over time, yielding nonstationary membership functions of the form
$$\mu_{\dot{A}}(x, c(t), \sigma) = e^{-(x-(c+kf(t)))^2/2\sigma^2}; \tag{4}$$
2) variation in width: only the standard deviation σ of the Gaussian (3) was varied over time, yielding nonstationary membership functions of the form
$$\mu_{\dot{A}}(x, c, \sigma(t)) = e^{-(x-c)^2/2(\sigma+kf(t))^2}. \tag{5}$$

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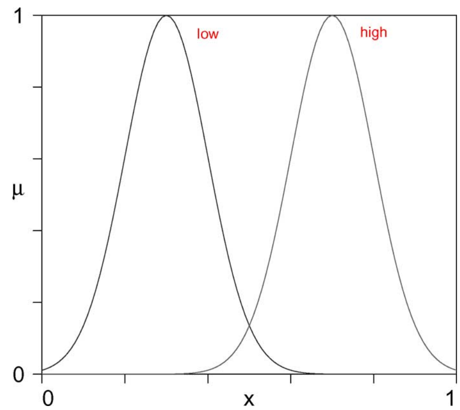

Fig. 6. Underlying type-1 Gaussian membership functions for the terms *low* and *high*, as used in the first case study.

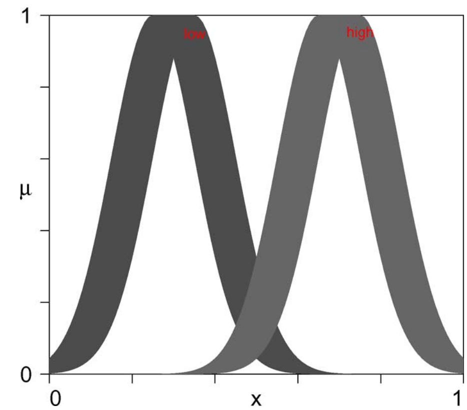

Fig. 7. Footprints of uncertainty of the interval type-2 fuzzy sets for *low* and *high*, for variation in location of Gaussian membership functions.

The following three different perturbation functions were used:

1) a random function with uniform distribution (R *runif*);
2) a random function with Gaussian distribution (R *rnorm*);
3) a sinusoidal function [(1) with $\omega = 127$];

and $k = 0.05$. The aforementioned first and third functions returned values in the range $[-1, 1]$, while the second one (the R *rnorm* function) returned real values sampled from the normalized Gaussian distribution.

Four different nonstationary FISs for each of these three perturbation functions were designed (12 nonstationary FISs in total). These were distinguished by the number of instantiations (time points) used to construct the nonstationary fuzzy sets, as the following:

1) 30 instantiations;
2) 100 instantiations;
3) 1000 instantiations;
4) 10 000 instantiations.

2) Type-2 FISs: Two interval type-2 FISs featuring the same inputs and outputs, and the same four rules as the earlier nonstationary FISs were designed. The footprints of uncertainty were created by deviating the parameters of (3). For variation in location, the lower and the upper bounds of the FOU were generated by

$$\mu_{\tilde{A}}(x, c, \sigma) = e^{-(x-(c\pm k))^2/2\sigma^2}.$$

For variation in width, the lower and the upper bounds of the FOU were generated by

$$\mu_{\tilde{A}}(x, c, \sigma) = e^{-(x-c)^2/2(\sigma\pm k)^2}.$$

Note that these formulas were obtained from (4) and (5), respectively, by setting $f(t) = \pm 1$, and that $k = 0.05$. This was purposefully chosen to establish a form of correspondence between the nonstationary and the interval type-2 FISs. The footprints of uncertainty of the interval type-2 fuzzy sets, for variation in location, are shown in Fig. 7.

3) Inference Processes: For each FIS, the inference processes were performed and the results were obtained as follows. For the nonstationary systems, the inference process was performed for the specified number of times (equal to the number of instantiations) to obtain the set of (type-1) output fuzzy sets, for each input combination. In due course, defuzzification was applied to each output set to obtain the standard center of gravity g. As a result, a set of the centers of gravity G was obtained (it is obvious that $|G| = |T|$). For uniform and sinusoidal perturbation functions (the range bounded to $[-1, +1]$), the minimum of G was taken as the lower bound, the maximum of G as the upper bound, and the arithmetic mean was taken as the mean. For Gaussian perturbation functions (unbounded range), the lower and the upper bounds were derived by $c_G \pm \sigma_G$, where c_G is the mean and σ_G is the standard deviation of G.

For the type-2 systems, the four combinations shown in Table III were presented to the inputs. For each input combination, the inference was performed using the rules given to obtain the type-2 output set. Karnik–Mendel type reduction was used to obtain the lower and the upper bounds of the center of gravity of the output. The mean of the output was taken as the average of the lower and upper bounds.

4) Results: The results obtained for variation in the center of the underlying Gaussian membership functions are given in Table IV, while the results obtained for variation in the standard deviation of the underlying Gaussian membership functions are given in Table V. Both the inputs to and the outputs of the NS1-GL-G30, NS1-GL-G10000, NS1-GL-U30, and NS1-GL-U10000 systems were chosen for further investigation. The instantiations of the inputs were obtained for all the aforementioned systems; they are shown in Fig. 8. Only for the NS1-GL-G10000 and NS1-GL-U10000 systems, the distributions of membership grades of the inputs over time for the term *low* and the values of x of 0.2 and 0.3 were obtained. As such distributions are obtained by taking a "vertical slice" through a nonstationary fuzzy set, they are in some way analogous to secondary membership functions of a type-2 fuzzy set. These distributions are shown in Fig. 9.

For the outputs, the instantiations were obtained for all the systems specified earlier; they are shown in Fig. 10. Only for the NS1-GL-G10000 and NS1-GL-U10000 systems, the distributions of membership grades of the outputs over time for the input combination of $(0.25, 0.25)$ and the values of x of 0.2 and 0.3 were obtained (Fig. 11). Finally, the distributions of the centers of gravity for the NS1-GL-G10000 and NS1-GL-U10000 systems, with input combinations of $(0.25, 0.25)$

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TABLE IV
LOWER BOUNDS, MEANS, AND UPPER BOUNDS FOR VARIATION IN CENTER OF UNDERLYING GAUSSIAN MEMBERSHIP FUNCTIONS

	Combination I $(0.25, 0.25)$			*Combination II* $(0.25, 0.75)$			*Combination III* $(0.75, 0.25)$			*Combination IV* $(0.75, 0.75)$		
FIS	*Lower*	*Mean*	*Upper*	*Lower*	*Mean*	*Upper*	*Lower*	*Mean*	*Upper*	*Lower*	*Mean*	*Upper*
IT2-GL	0.2385	0.3013	0.3640	0.6360	0.6987	0.7615	0.6360	0.6987	0.7615	0.2385	0.3013	0.3640
NS1-GL-U30	0.2538	0.2937	0.3488	0.6506	0.6989	0.7464	0.6506	0.6989	0.7468	0.2539	0.2938	0.3487
NS1-GL-U100	0.2525	0.3007	0.3491	0.6505	0.6963	0.7464	0.6504	0.6963	0.7468	0.2525	0.3007	0.3491
NS1-GL-U1000	0.2522	0.3018	0.3501	0.6501	0.7000	0.7481	0.6500	0.7000	0.7482	0.2523	0.3018	0.3501
NS1-GL-U10000	0.2519	0.3012	0.3502	0.6498	0.6989	0.7481	0.6499	0.6989	0.7482	0.2520	0.3012	0.3502
NS1-GL-G30	0.2522	0.3083	0.3644	0.6362	0.6915	0.7467	0.6363	0.6915	0.7468	0.2527	0.3084	0.3641
NS1-GL-G100	0.2635	0.3127	0.3619	0.6432	0.6905	0.7378	0.6430	0.6905	0.7379	0.2636	0.3128	0.3621
NS1-GL-G1000	0.2558	0.3045	0.3532	0.6485	0.6969	0.7454	0.6486	0.6970	0.7455	0.2560	0.3046	0.3533
NS1-GL-G10000	0.2534	0.3017	0.3500	0.6504	0.6984	0.7463	0.6504	0.6984	0.7463	0.2533	0.3017	0.3500
NS1-GL-S30	0.2518	0.3010	0.3501	0.6501	0.6989	0.7480	0.6500	0.6989	0.7478	0.2519	0.3011	0.3502
NS1-GL-S100	0.2518	0.3012	0.3502	0.6499	0.6993	0.7480	0.6498	0.6992	0.7478	0.2519	0.3012	0.3503
NS1-GL-S1000	0.2518	0.3008	0.3502	0.6499	0.6992	0.7480	0.6498	0.6991	0.7478	0.2519	0.3009	0.3503
NS1-GL-S10000	0.2518	0.3008	0.3502	0.6499	0.6991	0.7480	0.6498	0.6991	0.7478	0.2519	0.3008	0.3503

TABLE V
LOWER BOUNDS, MEANS, AND UPPER BOUNDS FOR VARIATION IN STANDARD DEVIATION OF UNDERLYING GAUSSIAN MEMBERSHIP FUNCTIONS

	Combination I $(0.25, 0.25)$			*Combination II* $(0.25, 0.75)$			*Combination III* $(0.75, 0.25)$			*Combination IV* $(0.75, 0.75)$		
FIS	*Lower*	*Mean*	*Upper*	*Lower*	*Mean*	*Upper*	*Lower*	*Mean*	*Upper*	*Lower*	*Mean*	*Upper*
IT2-GW	0.2056	0.3101	0.4146	0.5854	0.6899	0.7944	0.5854	0.6899	0.7944	0.2056	0.3101	0.4146
NS1-GW-U30	0.3000	0.3031	0.3096	0.6910	0.6955	0.7000	0.6876	0.6960	0.7000	0.3000	0.3039	0.3088
NS1-GW-U100	0.3000	0.3039	0.3106	0.6887	0.6962	0.7000	0.6809	0.6963	0.7000	0.3000	0.3035	0.3126
NS1-GW-U1000	0.3000	0.3038	0.3193	0.6822	0.6962	0.7000	0.6806	0.6962	0.7000	0.3000	0.3039	0.3181
NS1-GW-U10000	0.3000	0.3038	0.3193	0.6814	0.6962	0.7000	0.6806	0.6962	0.7000	0.3000	0.3038	0.3209
NS1-GW-G30	0.2959	0.3106	0.3253	0.6623	0.6829	0.7034	0.6170	0.6764	0.7358	0.2685	0.3261	0.3837
NS1-GW-G100	0.2638	0.3316	0.3993	0.6288	0.6772	0.7256	0.5789	0.6630	0.7470	0.2608	0.3312	0.4017
NS1-GW-G1000	0.2619	0.3353	0.4086	0.5986	0.6683	0.7379	0.5942	0.6669	0.7397	0.2623	0.3338	0.4053
NS1-GW-G10000	0.2608	0.3350	0.4092	0.5940	0.6661	0.7383	0.5894	0.6647	0.7400	0.2611	0.3359	0.4106
NS1-GW-S30	0.3000	0.3059	0.3158	0.6895	0.6951	0.7000	0.6886	0.6952	0.7000	0.3000	0.3039	0.3135
NS1-GW-S100	0.3000	0.3060	0.3158	0.6895	0.6951	0.7000	0.6886	0.6951	0.7000	0.3000	0.3039	0.3135
NS1-GW-S1000	0.3000	0.3061	0.3158	0.6895	0.6951	0.7000	0.6886	0.6952	0.7000	0.3000	0.3039	0.3136
NS1-GW-S10000	0.3000	0.3061	0.3158	0.6895	0.6951	0.7000	0.6886	0.6952	0.7000	0.3000	0.3039	0.3136

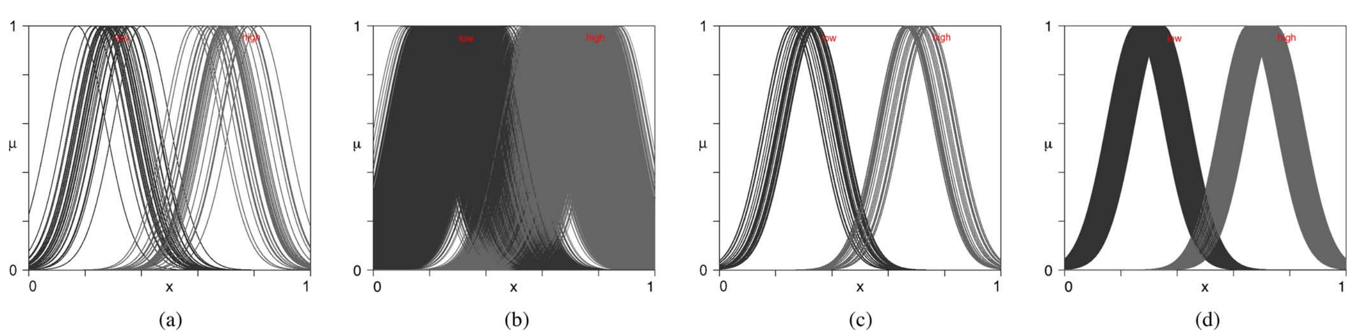

Fig. 8. Instantiations of selected nonstationary fuzzy inputs for Gaussian underlying membership functions. (a) GL-G30. (b) GL-G10000. (c) GL-U30. (d) GL-U10000.

and $(0.25, 0.75)$, were examined. These distributions are shown in Fig. 12.

C. Case Study II: Triangular Membership Functions

1) Nonstationary FISs: In the second case study, the same experiments as described earlier were repeated. However, this time triangular functions were used as the underlying membership functions. That is, the underlying membership functions were of the form

$$\mu_A(x,a,b,c) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{c-a}, & a < x \le c \\ \dfrac{b-x}{b-c}, & c < x < b \\ 0, & x \ge b \end{cases} \tag{6}$$

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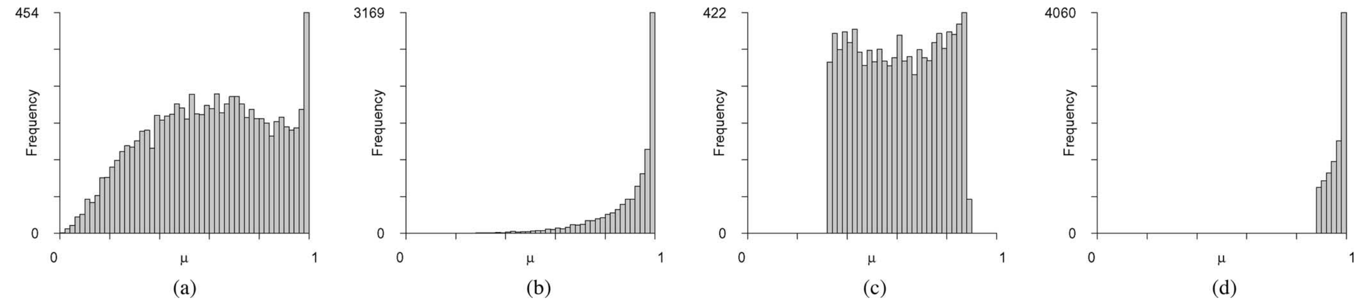

Fig. 9. Distributions of membership grades over time for the term *low* of selected nonstationary inputs. (a) GL-G10000 at $x = 0.20$. (b) GL-G10000 at $x = 0.30$. (c) GL-U10000 at $x = 0.20$. (d) GL-G10000 at $x = 0.30$.

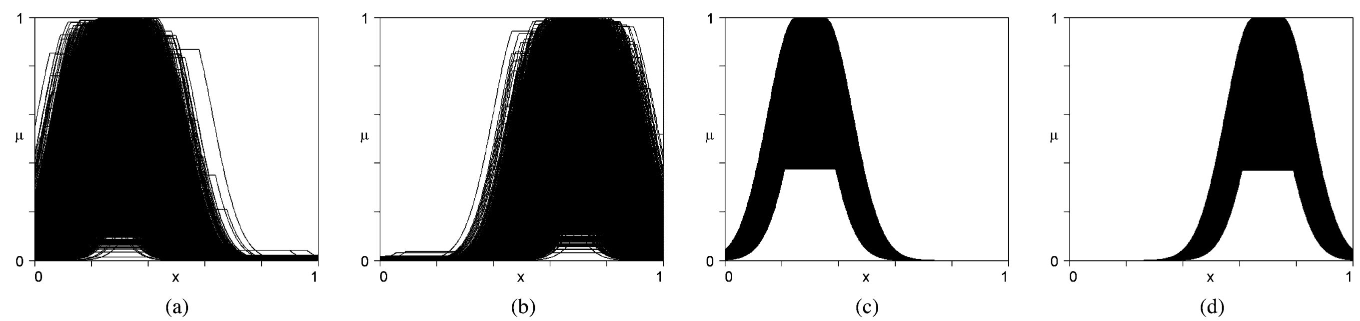

Fig. 10. Instantiations of selected nonstationary outputs for the specified input combinations. (a) GL-G10000 for $(0.25, 0.25)$. (b) GL-G10000 for $(0.25, 0.75)$. (c) GL-U10000 for $(0.25, 0.25)$. (d) GL-U10000 for $(0.25, 0.75)$.

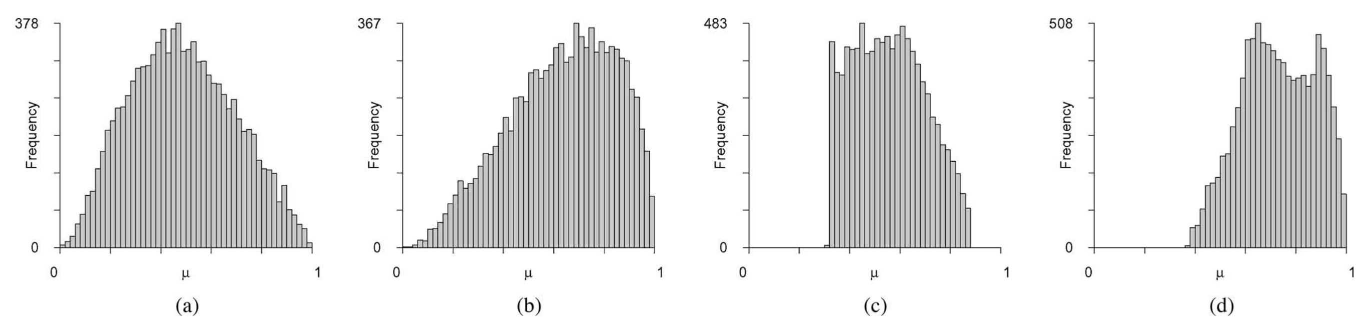

Fig. 11. Distributions of membership grades over time for selected nonstationary outputs with the input combination $(0.25, 0.25)$. (a) GL-G10000 at $x = 0.20$. (b) GL-G10000 at $x = 0.30$. (c) GL-U10000 at $x = 0.20$. (d) GL-U10000 at $x = 0.30$.

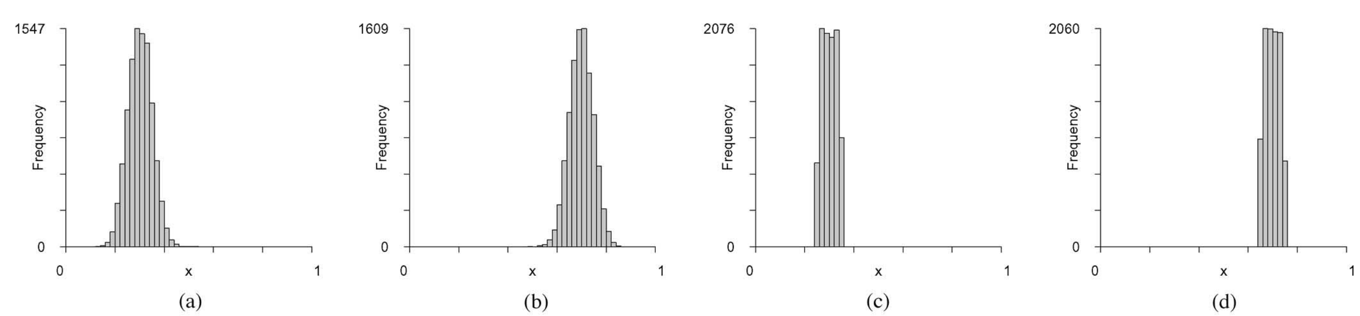

Fig. 12. Distributions of the centers of gravity for selected nonstationary outputs with the specified input combinations. (a) GL-G10000 for $(0.25, 0.25)$. (b) GL-G10000 for $(0.25, 0.75)$. (c) GL-U10000 for $(0.25, 0.25)$. (d) GL-U10000 for $(0.25, 0.75)$.

where a denotes the left-hand base-point of the triangle, b denotes the right-hand base-point, and c denotes the center of the triangle. The *low* underlying membership function had $a = 0.1$, $b = 0.5$, and $c = 0.3$, and the *high* underlying membership function had $a = 0.5$, $b = 0.9$, and $c = 0.7$. These functions are shown in Fig. 13.

Only one form of nonstationarity was implemented in this case study—variation in location, yielding nonstationary

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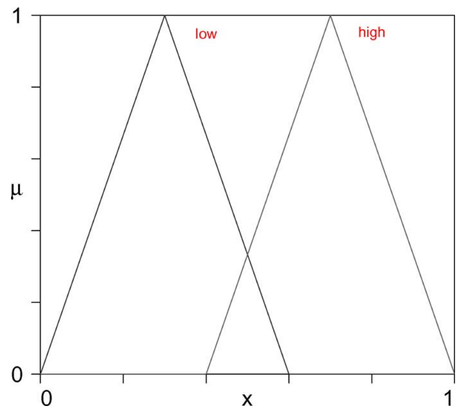

Fig. 13. Underlying type-1 triangular membership functions for the terms *low* and *high*, as used in the second case study.

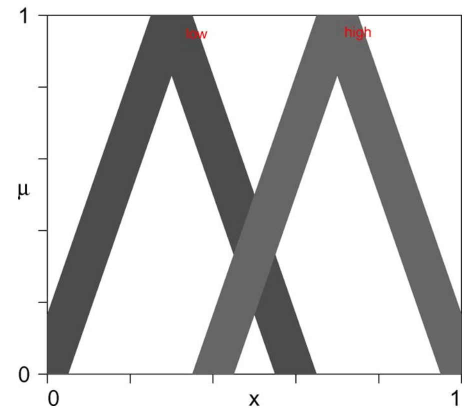

Fig. 14. Footprints of uncertainty of the interval type-2 fuzzy sets for *low* and *high*, for variation in location of triangular membership functions.

membership functions of the form

$$\mu_{\dot{A}}(x, a(t), b(t), c(t)) = \begin{cases} 0, & x - kf(t) \le a \\ \dfrac{x - (a + kf(t))}{c - a}, & a < x - kf(t) \le c \\ \dfrac{(b + kf(t)) - x}{b - c}, & c < x - kf(t) < b \\ 0, & x - kf(t) \ge b \end{cases}$$

so that the whole triangle was shifted left or right over time by the amount $kf(t)$.

The following same three perturbations functions were used:

1) a random function with uniform distribution (R *runif*);
2) a random function with Gaussian distribution (R *rnorm*);
3) a sinusoidal function [(1) with $\omega = 127$];

and $k = 0.05$.

Again, four different nonstationary systems for each of these perturbation function were created, with 30, 100, 1000 and 10 000 instantiations.

2) Type-2 FIS: The interval type-2 system was generated using the same principles as described in Section V-B.2. So, the FOU was bounded by

$$\mu_{\tilde{A}}(x, a, b, c) = \begin{cases} 0, & x - k \le a \\ \dfrac{x - (a + k)}{c - a}, & a < x - k \le c \\ \dfrac{(b + k) - x}{b - c}, & c < x - k < b \\ 0, & x - k \ge b \end{cases}$$

and

$$\mu_{\tilde{A}}(x, a, b, c) = \begin{cases} 0, & x + k \le a \\ \dfrac{x - (a - k)}{c - a}, & a < x + k \le c \\ \dfrac{(b - k) - x}{b - c}, & c < x + k < b \\ 0, & x + k \ge b \end{cases}$$

again with $k = 0.05$. The footprints of uncertainty of the interval type-2 fuzzy sets are shown in Fig. 14.

3) Inference Processes: The inference was performed using the 13 systems (12 nonstationary FISs and the type-2 FIS), and the methodology as described earlier was used to derive the results for each FIS.

4) Results: The results obtained for variation in location of the underlying triangular membership functions are shown in Table VI. Both the inputs to and the outputs of the NS1-TL-G30, NS1-TL-G10000, NS1-TL-U30, and NS1-TL-U10000 systems were chosen for further investigation. The instantiations of the inputs were obtained for all the aforementioned systems; they are shown in Fig. 15. Only for the NS1-TL-G10000 and NS1-TL-U10000 systems, the distributions of membership grades of the inputs over time for the term *low* and the values of x of 0.2 and 0.3 were obtained. These distributions are shown in Fig. 16.

For the outputs, the instantiations were obtained for all the systems specified earlier; they are shown in Fig. 17. Only for the NS1-TL-G10000 and NS1-TL-U10000 systems, the distributions of membership grades of the outputs over time for the input combination of $(0.25, 0.25)$ and the values of x of 0.2 and 0.3 were obtained (Fig. 18). Finally, the distributions of the centers of gravity, only for the NS1-TL-G10000 and NS1-TL-U10000 systems, with input combinations of $(0.25, 0.25)$ and $(0.25, 0.75)$, were examined. These distributions are shown in Fig. 19.

VI. Discussion

We have defined a new concept that we term a *nonstationary fuzzy set*. These have been created with the specific intention of modeling the variation (over time) of opinion, and they formalize the novel concept that we had previously introduced [15] to model the variation in expert opinion. While superficially similar to type-2 fuzzy sets in some regards, nonstationary fuzzy sets possess some important distinguishing features. The first is that a nonstationary fuzzy set is, effectively, a collection of type-1 fuzzy sets in which there is an explicit, defined, relationship between the fuzzy sets. Specifically, each of the instantiations (individual type-1 sets) is derived by a perturbation of (making a small change to) a single underlying membership function. While each instantiation is somewhat reminiscent of an embedded type-1 set of a type-2 fuzzy set, there is *not* a direct

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TABLE VI
LOWER BOUNDS, MEANS, AND UPPER BOUNDS FOR VARIATION IN LOCATION OF THE UNDERLYING TRIANGULAR MEMBERSHIP FUNCTIONS

	Combination I $(0.25, 0.25)$			*Combination II* $(0.25, 0.75)$			*Combination III* $(0.75, 0.25)$			*Combination IV* $(0.75, 0.75)$		
FIS	*Lower*	*Mean*	*Upper*	*Lower*	*Mean*	*Upper*	*Lower*	*Mean*	*Upper*	*Lower*	*Mean*	*Upper*
IT2-TL	0.2440	0.3021	0.3603	0.6397	0.6979	0.7560	0.6397	0.6979	0.7560	0.2440	0.3021	0.3603
NS1-TL-U30	0.2558	0.2938	0.3486	0.6508	0.6988	0.7444	0.6508	0.6988	0.7450	0.2559	0.2938	0.3486
NS1-TL-U100	0.2544	0.3006	0.3489	0.6506	0.6963	0.7444	0.6506	0.6964	0.7450	0.2546	0.3006	0.3489
NS1-TL-U1000	0.2542	0.3017	0.3500	0.6502	0.7001	0.7461	0.6502	0.7001	0.7461	0.2542	0.3017	0.3500
NS1-TL-U10000	0.2538	0.3011	0.3500	0.6500	0.6990	0.7463	0.6500	0.6990	0.7463	0.2539	0.3011	0.3500
NS1-TL-G30	0.2557	0.3091	0.3626	0.6371	0.6908	0.7446	0.6371	0.6908	0.7444	0.2567	0.3094	0.3621
NS1-TL-G100	0.2652	0.3130	0.3608	0.6441	0.6902	0.7364	0.6439	0.6899	0.7359	0.2653	0.3134	0.3616
NS1-TL-G1000	0.2579	0.3051	0.3522	0.6494	0.6964	0.7434	0.6496	0.6965	0.7434	0.2580	0.3051	0.3522
NS1-TL-G10000	0.2555	0.3022	0.3490	0.6515	0.6979	0.7443	0.6515	0.6979	0.7442	0.2554	0.3022	0.3490
NS1-TL-S30	0.2537	0.3012	0.3499	0.6502	0.6988	0.7461	0.6502	0.6987	0.7456	0.2540	0.3013	0.3499
NS1-TL-S100	0.2537	0.3013	0.3500	0.6500	0.6991	0.7461	0.6500	0.6990	0.7456	0.2540	0.3014	0.3500
NS1-TL-S1000	0.2537	0.3010	0.3500	0.6500	0.6990	0.7461	0.6500	0.6989	0.7456	0.2540	0.3010	0.3500
NS1-TL-S10000	0.2537	0.3009	0.3500	0.6500	0.6990	0.7461	0.6500	0.6989	0.7456	0.2540	0.3010	0.3500

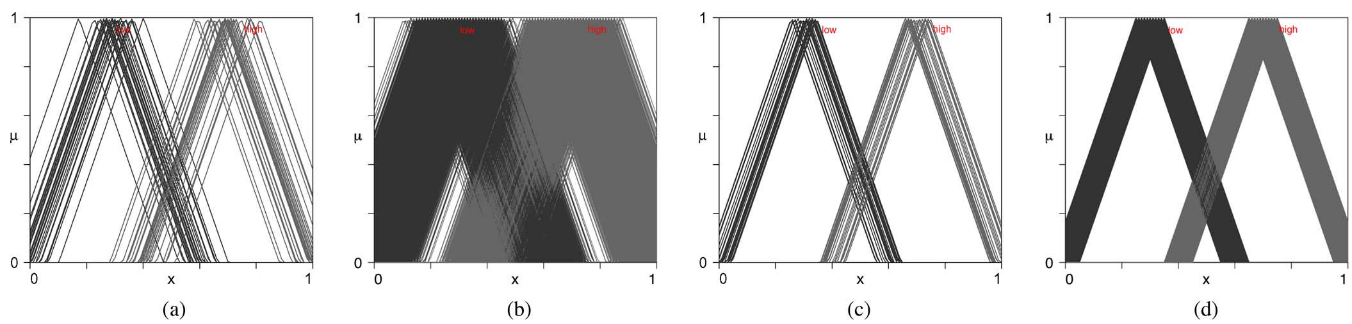

Fig. 15. Instantiations of selected nonstationary fuzzy inputs for triangular underlying membership functions. (a) TL-G30. (b) TL-G10000. (c) TL-U30. (d) TL-U10000.

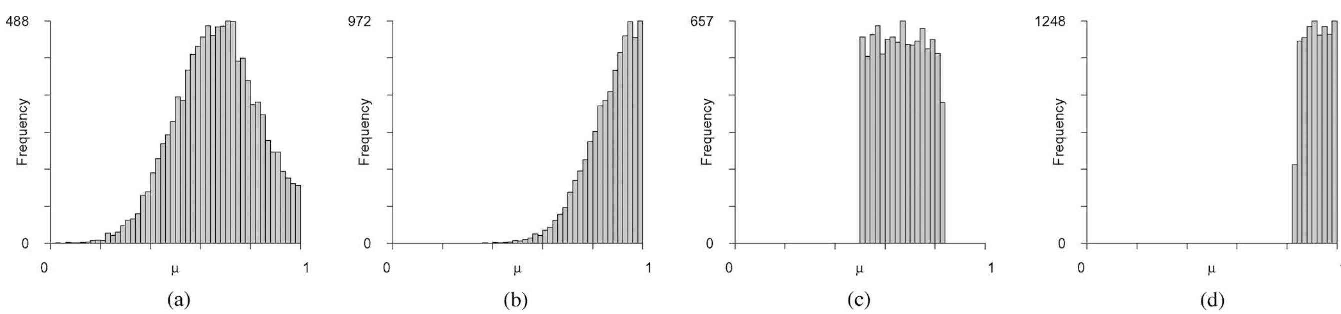

Fig. 16. Distributions of membership grades over time for the term *low* of selected nonstationary inputs. (a) TL-G10000 at $x = 0.20$. (b) TL-G10000 at $x = 0.30$. (c) TL-U10000 at $x = 0.20$. (d) TL-U10000 at $x = 0.30$.

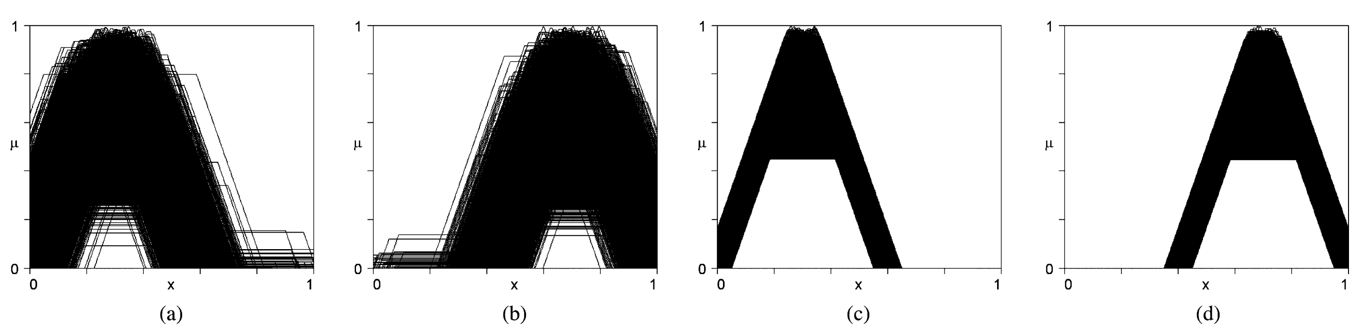

Fig. 17. Instantiations of selected nonstationary outputs for the specified input combinations. (a) TL-G10000 for $(0.25, 0.25)$. (b) TL-G10000 for $(0.25, 0.75)$. (c) TL-U10000 for $(0.25, 0.25)$. (d) TL-U10000 for $(0.25, 0.75)$.

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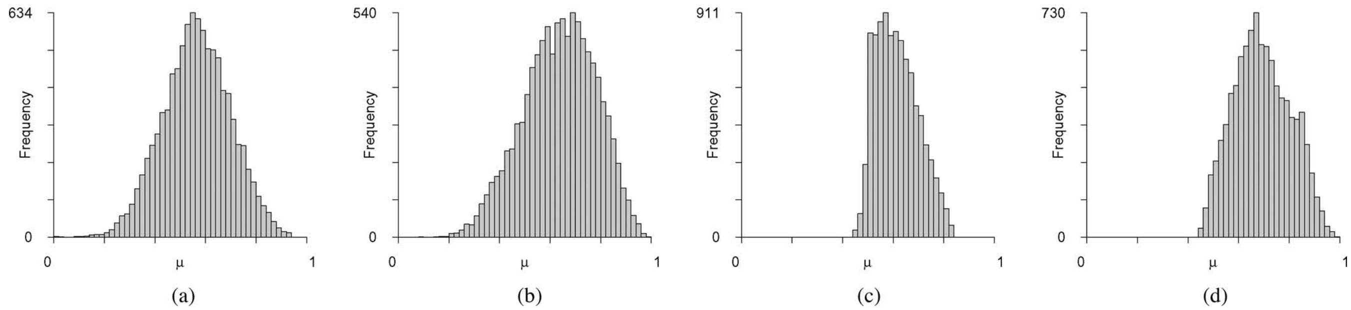

Fig. 18. Distributions of membership grades over time for selected nonstationary outputs with the input combination $(0.25, 0.25)$. (a) TL-G10000 at $x = 0.20$. (b) TL-G10000 at $x = 0.30$. (c) TL-U10000 at $x = 0.20$. (d) TL-U10000 and $x = 0.30$.

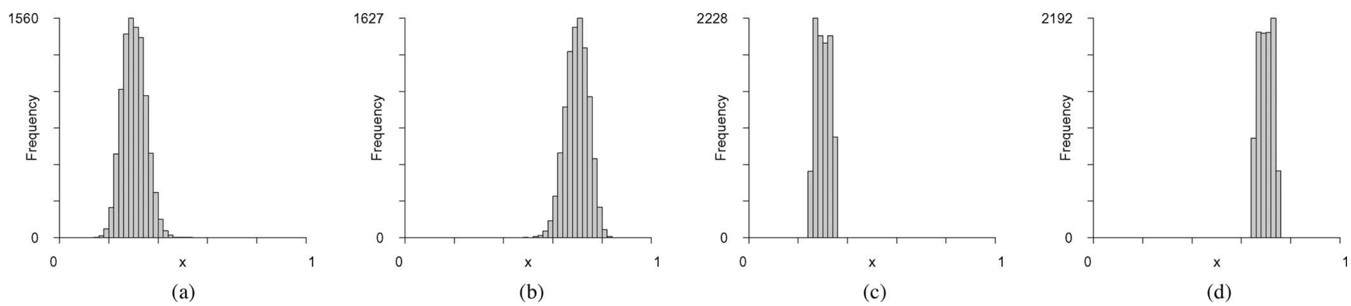

Fig. 19. Distributions of the centers of gravity for selected nonstationary outputs with the specified input combinations. (a) TL-G10000 for $(0.25, 0.25)$. (b) TL-G10000 for $(0.25, 0.75)$. (c) TL-U10000 for $(0.25, 0.25)$. (d) TL-U10000 for $(0.25, 0.75)$.

correspondence between these two concepts. In the FOU shown in Fig. 7, for example, there are very many embedded type-1 fuzzy sets that are non-Gaussian. As a corollary, it is also possible to view a standard type-1 fuzzy set as being a specific case of a nonstationary fuzzy set, either as a single instantiation or as repeated instantiations of the underlying set with no perturbation.

Second, a nonstationary fuzzy set does not have secondary membership functions. Hence, there is no direct equivalent to the embedded type-2 sets of a type-2 fuzzy set. Similarly, there are no secondary membership grades. While it is true that distributions of membership grades across "vertical slices" can be observed (as in Figs. 9, 11, 16, and 18), such distributions are still not, formally, the same as secondary membership functions. The form of these distributions is governed by the following three factors:

1) the underlying membership function [compare Figs. 9(a) and 16(a)];
2) the parameter being varied (obvious from consideration of the different distributions that would be obtained at $x = 0.25$ for variation in center and variation in standard deviation of a Gaussian with a center of 0.25); and
3) the perturbation function [compare Figs. 9(a) and 16(c)].

If, for a specific application, it is decided *a priori* that a specific secondary membership function is required for the inputs, then it may well be that type-2 fuzzy sets offer the most suitable modeling technique. Having said this, the form of secondary membership function of the output type-2 fuzzy set(s) produced as a result of the general type-2 inference is not known *a priori*.

Third, the inference process is quite different. The crucial point is that, at any instant of time, a nonstationary fuzzy set is (instantiates) a type-1 fuzzy set. Hence, the nonstationary inference is just a repeated type-1 inference (albeit with slightly different type-1 sets at each time instant). In contrast, type-2 inference involves passing type-2 fuzzy sets through the process, resulting in type-2 output sets that require type reduction prior to defuzzification. It is well known that, while interval type-2 fuzzy sets permit tractable inference and type reduction, the use of general type-2 fuzzy sets renders this intractable (although some advances have been made recently in providing approximations, e.g., [18]). On the other hand, altering the perturbation function within a nonstationary FIS has no effect on the difficulty of the inferencing process. This observation means that if the correspondence can be more formally established, nonstationary systems featuring nonuniformly distributed perturbations *may* allow approximations of the general type-2 fuzzy inference to be performed.

While the main point of carrying out the case studies was to demonstrate the ease of performing nonstationary inference, regardless of the form of underlying membership functions, parameter variation, and perturbation functions, the examination of the results highlights some interesting observations. For variation in center of Gaussian underlying membership functions, the output interval of the type-2 FIS for input combination I $(0.25, 0.25)$ is $[0.2385, 0.3640]$ with mean 0.3013. For the nonstationary FIS featuring, for example, uniform perturbation function and instantiated 10 000 times, the corresponding interval is [0.2519, 0.3502], with mean 0.3012; that is, the mean is very similar, but the interval is *narrower*. While it is clear that nonstationary fuzzy sets are fundamentally different from type-2 sets, it is also clear that, in some sense, a nonstationary set resembles a type-2 set in which the union of all possible instantiations of a nonstationary fuzzy set is similar to the FOU of a type-2 set. Note the striking similarity between the FOUs

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TABLE VII
SELECTED SET THEORETIC LAWS SATISFIED BY NONSTATIONARY FUZZY SETS

Involution	$\bar{\bar{\dot{A}}} = \dot{A}$	
	t-conorm	t-norm
Commutativity	$\dot{A} \cup \dot{B} = \dot{B} \cup \dot{A}$	$\dot{A} \cap \dot{B} = \dot{B} \cap \dot{A}$
Associativity	$\dot{A} \cup (\dot{B} \cup \dot{C}) = (\dot{A} \cup \dot{B}) \cup \dot{C}$	$\dot{A} \cap (\dot{B} \cap \dot{C}) = (\dot{A} \cap \dot{B}) \cap \dot{C}$
Distributivity	$\dot{A} \cap (\dot{B} \cup \dot{C}) = (\dot{A} \cap \dot{B}) \cup (\dot{A} \cap \dot{C})$	$\dot{A} \cup (\dot{B} \cap \dot{C}) = (\dot{A} \cup \dot{B}) \cap (\dot{A} \cup \dot{C})$
Idempotence	$\dot{A} \cup \dot{A} = \dot{A}$ for max only	$\dot{A} \cap \dot{A} = \dot{A}$ for min only

in Figs. 7 and 14, and the instantiations in Figs. 8(d) and 15(d), respectively.

To reiterate, it is a straightforward task to perform the nonstationary inference and to observe the distributions of membership grades across vertical slices of the resultant output variables. It is evident from Figs. 11 and 18 that these observed distributions are nonuniform. While these are *not* secondary membership functions and this is obviously *not the same* as the general type-2 inference, it is apparent that nonstationary fuzzy sets permit inference with fuzzy sets that do not have fixed, precise (i.e., type-1) membership functions. Furthermore, this inference directly results in a distribution of centers of gravity, which may be nonuniform (see Figs. 12 and 19). With the interval type-2 inference, the type-reduction is required to obtain just the lower and upper bounds of the center of gravity (and this currently requires an iterative approximation method).

Finally, the relationships between nonstationary fuzzy sets and other related concepts such as fuzzy random sets [19], fuzzy random variables [20], and more recent developments such as type-2 fuzzy hidden Markov models [21] remain to be explored and established.

In summary, more research needs to be done on nonstationary fuzzy sets before any definitive claims can be made in regard to their more general usefulness and to the correspondence between them and type-2 sets. We conclude that, even at this early stage, nonstationary fuzzy sets are a novel complementary addition to the range of fuzzy methods. Our research on understanding and modeling the dynamics of variation in human decision making is ongoing, and issues surrounding the use of nonstationary fuzzy sets will be further explored.

APPENDIX I
PROOF OF PROPERTIES OF NONSTATIONARY FUZZY SETS

This section is dedicated to the proofs of fundamental properties of the nonstationary fuzzy set operators defined earlier. These proofs are derived directly from Zadeh's proofs for standard type-1 fuzzy sets; we include them for completeness. Table VII summarizes the set theoretic laws that are satisfied by nonstationary fuzzy sets.

Let us consider nonstationary fuzzy sets $\dot{A}$, $\dot{B}$, and $\dot{C}$ as

$$\dot{A} = \int_{t\in T}\int_{x\in X} \mu_{\dot{A}}(t,x)/x/t$$

$$\dot{B} = \int_{t\in T}\int_{x\in X} \mu_{\dot{B}}(t,x)/x/t$$

and

$$\dot{C} = \int_{t\in T}\int_{x\in X} \mu_{\dot{C}}(t,x)/x/t.$$

Note that, for the sake of brevity in the later formulas, whenever we use the nonstationary union, intersection, or complement operators defined in Section IV, we omit $\forall(t,x) \in T \times X$.

A. Involution

Let us consider the complement of $\dot{A}$, $\bar{\dot{A}}$ as

$$\bar{\dot{A}} = \int_{t\in T}\int_{x\in X} \mu_{\bar{\dot{A}}}(t,x)/x/t.$$

By the definition of the standard complement operator for nonstationary fuzzy sets, we have

$$\mu_{\bar{\dot{A}}}(t,x) = 1 - \mu_{\dot{A}}(t,x).$$

Thus, the complement of $\bar{\dot{A}}$ can be expressed as

$$\bar{\bar{\dot{A}}} = \int_{t\in T}\int_{x\in X} \mu_{\bar{\bar{\dot{A}}}}(t,x)/x/t$$

where

$$\mu_{\bar{\bar{\dot{A}}}}(t,x) = 1 - \mu_{\bar{\dot{A}}}(t,x).$$

Since $\mu_{\bar{\dot{A}}}(t,x) = 1 - \mu_{\dot{A}}(t,x)$, we obtain

$$\mu_{\bar{\bar{\dot{A}}}}(t,x) = 1 - (1 - \mu_{\dot{A}}(t,x)).$$

It follows that

$$\mu_{\bar{\bar{\dot{A}}}}(t,x) = \mu_{\dot{A}}(t,x).$$

B. Commutativity

1) Union: By the definition of the union operator for nonstationary fuzzy sets, we have

$$\mu_{\dot{A}\cup\dot{B}}(t,x) = \mu_{\dot{A}}(t,x) \oplus \mu_{\dot{B}}(t,x).$$

As t-conorm operators are commutative, we know that

$$\mu_{\dot{A}}(t,x) \oplus \mu_{\dot{B}}(t,x) = \mu_{\dot{B}}(t,x) \oplus \mu_{\dot{A}}(t,x).$$

Again, by the definition

$$\mu_{\dot{B}\cup\dot{A}}(t,x) = \mu_{\dot{B}}(t,x) \oplus \mu_{\dot{A}}(t,x)$$

thus

$$\mu_{\dot{A}\cup\dot{B}}(t,x) = \mu_{\dot{B}\cup\dot{A}}(t,x).$$

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2) Intersection: By the definition of the intersection operator for nonstationary fuzzy sets, we have

$$\mu_{\dot{A}\cap\dot{B}}(t,x) = \mu_{\dot{A}}(t,x) \otimes \mu_{\dot{B}}(t,x).$$

As t-norm operators are commutative, we know that

$$\mu_{\dot{A}}(t,x) \otimes \mu_{\dot{B}}(t,x) = \mu_{\dot{B}}(t,x) \otimes \mu_{\dot{A}}(t,x).$$

Again, by the definition

$$\mu_{\dot{B}\cap\dot{A}}(t,x) = \mu_{\dot{B}}(t,x) \otimes \mu_{\dot{A}}(t,x)$$

thus

$$\mu_{\dot{A}\cap\dot{B}}(t,x) = \mu_{\dot{B}\cap\dot{A}}(t,x).$$

C. Associativity

1) Union: By the definition of the union operator for nonstationary fuzzy sets, we have

$$\mu_{\dot{A}\cup(\dot{B}\cup\dot{C})}(t,x) = \mu_{\dot{A}}(t,x) \oplus (\mu_{\dot{B}}(t,x) \oplus \mu_{\dot{C}}(t,x)).$$

As t-conorm operators are associative, we know that

$$\mu_{\dot{A}}(t,x) \oplus (\mu_{\dot{B}}(t,x) \oplus \mu_{\dot{C}}(t,x)) = (\mu_{\dot{A}}(t,x) \oplus \mu_{\dot{B}}(t,x)) \oplus \mu_{\dot{C}}(t,x).$$

Again, by the definition

$$\mu_{(\dot{A}\cup\dot{B})\cup\dot{C}}(t,x) = (\mu_{\dot{A}}(t,x) \oplus \mu_{\dot{B}}(t,x)) \oplus \mu_{\dot{C}}(t,x)$$

thus

$$\mu_{\dot{A}\cup(\dot{B}\cup\dot{C})}(t,x) = \mu_{(\dot{A}\cup\dot{B})\cup\dot{C}}(t,x).$$

2) Intersection: The proof is similar to that of the previous, substituting $\cap$ for $\cup$, t-norm for t-conorm, and $\otimes$ for $\oplus$.

D. Distributivity

The proofs of distributivity for nonstationary union and intersection follow exactly the form of those given earlier for associativity, and so were omitted for brevity.

E. Idempotence

1) Union: It is well known that only the maximum t-conorm is idempotent. Thus, by the definition of the union operator for nonstationary fuzzy sets, using the maximum t-conorm, we obtain

$$\mu_{\dot{A}\cup\dot{A}}(t,x) = \max(\mu_{\dot{A}}(t,x), \mu_{\dot{A}}(t,x)).$$

As the $\max$ operator is idempotent, we know that

$$\max(\mu_{\dot{A}}(t,x), \mu_{\dot{A}}(t,x)) = \mu_{\dot{A}}(t,x)$$

and so

$$\mu_{\dot{A}\cup\dot{A}}(t,x) = \mu_{\dot{A}}(t,x).$$

2) Intersection: As aforementioned, this is omitted for brevity.

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