New Concepts Related to Non-Stationary Fuzzy Sets

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***Abstract*— In this paper, formal definitions of the concepts relevant to non-stationary fuzzy sets are provided, as well as definitions of the basic non-stationary fuzzy set operators with proofs of selected properties of these operators. Among the novel terms introduced are the *footprint of instantiations*, the *domain of instantiations* and the *temporal histogram*. Further, we discuss the correspondence between non-stationary and type-2 fuzzy sets, and make the first attempt at proposing a set of comparable terms.**

I. Introduction

Type-2 fuzzy sets were introduced by Zadeh, in his seminal paper of 1975 [1], in order to permit '*fuzzy sets in which the grades of membership are specified in linguistic terms*'. Unfortunately, their use in practice was limited due to the significant amount of computational complexity involved in their implementation. More recently, type-2 fuzzy sets received renewed interest, mainly due to the efforts of Mendel [2], but also, possibly, due to the increases in computational power. Mendel established a set of terms to be used when working with type-2 fuzzy sets and, in particular, introduced the concept known as the *footprint of uncertainty* [3] which provides a useful verbal and graphical description of the uncertainty captured by any given type-2 fuzzy set. As a consequence, many publications on the theory and applications of type-2 fuzzy sets appeared (for example, [4] and [5]).

While type-2 fuzzy sets capture the concept of uncertainty in membership functions by introducing a range of membership values associated with each value of the base variable, they do not capture the notion of *variability* in that a type-2 fuzzy inference system (FIS) will always produce the same output(s) given the same input(s). On the other hand, it is well accepted that all humans, including 'experts', exhibit variation in their decision making. Variation may occur among the decisions of a panel of human experts (inter-expert variability), as well as in the decisions of an individual expert over time (intra-expert variability).

This motivated Garibaldi *et al.* to investigate the incorporation of variability into decision making in the context of fuzzy expert systems in a medical domain [6], [7], [8] and [9]. Consequently, Garibaldi proposed the notion of *non-deterministic fuzzy reasoning* [10] in which variability is introduced into the membership functions of a fuzzy system through the use of random alterations to the parameters of these functions. This notion was extended and formalised through the introduction of *non-stationary fuzzy sets* [11].

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In this paper, in Section II, a formal definition of a non-stationary fuzzy set is provided. The layout of Section III, containing the notions relevant to non-stationary fuzzy sets (including what we term the *footprint of instantiations* and the *domain of instantiations*), follows closely that of Section II in [2] and [12]. Definitions of standard operations applied to non-stationary fuzzy sets are provided in Section IV, and formal proofs of selected properties of these operations are given in Section V. In Section VI, we make preliminary comments on the parallels between non-stationary and type-2 fuzzy sets. Section VII concludes the paper and outlines the future work.

II. Non-stationary Fuzzy Sets

In this section, non-stationary fuzzy sets are defined. The definition is supported by an example depicting relationships between the introduced concepts.

A. Non-stationary Fuzzy Set Definition

Let A denote a fuzzy set of a universe of discourse X characterised by a membership function μ_A. Let T be a set of time points t_i (possibly infinite) and $f : T \rightarrow \Re$ denote a *perturbation function*.

Definition 1: A non-stationary fuzzy set $\dot{A}$ of the universe of discourse X is characterised by a *non-stationary membership function* $\mu_{\dot{A}} : T \times X \rightarrow [0,1]$ which associates each element (t,x) of $T \times X$ with a time-specific variation of $\mu_A(x)$. The non-stationary fuzzy set $\dot{A}$ is denoted by:

$$\dot{A} = \int_{t \in T} \int_{x \in X} \mu_{\dot{A}}(t,x)/x/t.$$

However, an additional restriction is imposed on $\mu_{\dot{A}}$. To formulate it in a coherent and precise way, let us first notice that $\mu_A(x)$ can be expressed as $\mu_A(x, p_1, \ldots, p_m)$, where $p_1, \ldots, p_m$ denote the parameters of $\mu_A(x)$. Now, we require that:

$$\mu_{\dot{A}}(t,x) = \mu_A(x, p_1(t), \ldots, p_m(t)),$$

where $p_i(t) = p_i + k_i f_i(t)$ and $i = 1, \ldots, m$. In this way, each parameter is varied in time by a perturbation function multiplied by a constant. ■

This definition establishes a relationship between standard and non-stationary fuzzy sets. Specifically, for a given standard fuzzy set A and a set of time points T, a non-stationary fuzzy set $\dot{A}$ is a set of duplicates of A varied over time. We term a time duplicate of A an *instantiation* and denote it by $\dot{A}_t$. Thus, at any given moment of time $t \in T$ the non-stationary fuzzy set $\dot{A}$ instantiates the standard fuzzy set $\dot{A}_t$. We term the standard fuzzy set A the *underlying* fuzzy set and its associated membership function, $\mu_A(x)$, the *underlying* membership function.

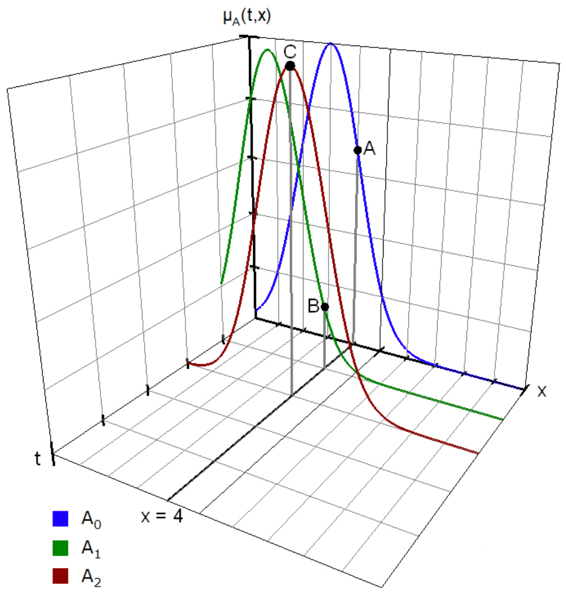

Fig. 1. A 3-D image of the non-stationary fuzzy set $\dot{A}$ from *Example 1*.

Example 1: As an example, let us consider the linguistic variable *average* representing the opinion of a domain expert on what will constitute the average change in the global temperature by the year 2100 (compared to the year 2005). Let us assume that the expert is asked for an opinion once a year. It seems convincing that the expert's opinion might change on a yearly basis (perhaps based on a new evidence provided by research efforts or, perhaps, by the latest fashion, 'fad' or emotional state).

The non-stationary fuzzy set $\dot{A}$ representing the opinion of the expert in the years 2005 – 2007, incorporating variability in the underlying membership function over time, might be represented, for example, by a Gaussian membership function where the centre parameter is a function of time:

$$\dot{A} = \int_{t \in T} \int_{x \in X} e^{-\frac{(x-c(t))^2}{2\sigma^2}} / x / t.$$

Let us assume that the universe of discourse X is the interval $[0, 10]$, with x interpreted as the change in the global temperature (in K or °C), and that the underlying membership function $\mu_A(x)$ is:

$$\mu_A(x) = e^{-\frac{(x-3)^2}{2(1.125^2)}}.$$

There are three instantiations, corresponding to the years 2005 ($\dot{A}_0$), 2006 ($\dot{A}_1$) and 2007 ($\dot{A}_2$), and $T = \{0, 1, 2\}$. Let us assume that $k_c = 1$ and that the perturbation function $f_c(t)$ is defined as:

$$f_c(t) = \begin{cases} 0 & \text{if } t = 0 \\ (-1)^t & \text{otherwise,} \end{cases}$$

so that $c(t) = 3 + f_c(t)$. The resulting non-stationary fuzzy set $\dot{A}$ is depicted in Fig 1. ■

B. Perturbation Functions

The original intention behind non-stationary fuzzy sets was to capture the notion of minor variations in a membership function corresponding to subtle differences in opinion over time. Thus, the term 'perturbation function' was deliberately chosen to imply that parameter changes induced by a function are 'small' or, more precisely, that parameter changes induce 'small' alterations in $\mu_A(x)$. This term was also deliberately chosen to imply that there is no permanent drift or change in the membership function, that might be characteristic of learning.

There are many ways in which an opinion may vary over time. Three main alternative forms of non-stationarity which might be more useful in practice are introduced, in a formal way, below.

- Variation in location:

$$\forall_{t \in T} \ \mu_{\dot{A}_t}(x) = \mu_A(x + c(t)),$$

  where $c(t)$ is a constant for any given $t \in T$. Thus, the membership function of the underlying fuzzy set is shifted, as a whole, left ($c_\alpha(t) > 0$) or right ($c_\alpha(t) < 0$) by small amounts along the universe of discourse.
- Variation in width:

$$\forall_{t \in T}, \forall_{\alpha \in [0,1]} \ |\dot{A}_{t,\alpha+}| = |A_{\alpha+}| + c_\alpha(t),$$

  where $c_\alpha(t)$ is a constant for any given $t \in T$. In this case, the cardinalities of all strong α-cuts of the underlying fuzzy set are increased ($c_\alpha(t) > 0$) or decreased ($c_\alpha(t) < 0$) by small amounts.
- Noise variation:

$$\forall_{t \in T} \ \mu_{\dot{A}_t}(x) = \mu_A(x) + \epsilon(t).$$

  Thus, a small amount of vertical shift is incorporated into the membership function.

III. Definitions of Related Concepts

In this Section, we provide formal definitions of the concepts related to non-stationary fuzzy sets. The aim of doing so is to establish a vocabulary of useful terms to be used when working with non-stationary fuzzy sets.

Definition 2: For each $x \in X$ the set:

$$G_x = \{(t, \mu_{\dot{A}}(t, x))\},$$

where t ranges over T, defines the *temporal membership function* $f_x : T \to \{\mu_{\dot{A}}(t, x)\}$. The set G_x will be called the *graph* of f_x.

Example 2: Let us recall the non-stationary fuzzy set from *Example 1*, depicted in Fig. 1. For $x = 4$ we have:

$$G_x = G_4 = \{(0, A), (1, B), (2, C)\}$$

and

$$f_x = f_4 : \{0, 1, 2\} \to \{A, B, C\}.$$

Definition 3: The set R_x of the values of f_x is termed the *temporal membership range*. Thus, $R_x = \{\mu_{\dot{A}}(t, x)\}$,

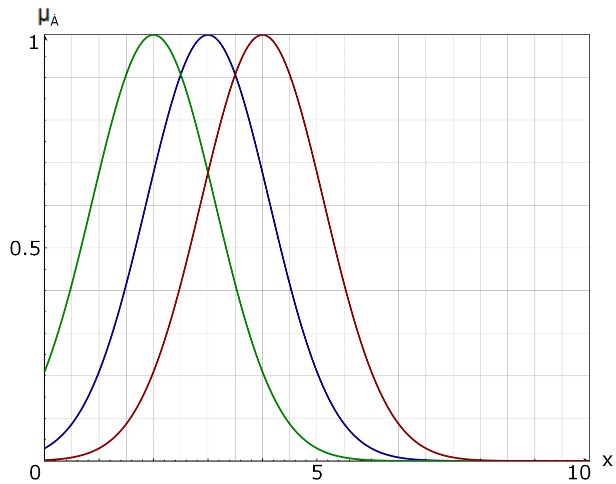

Fig. 2. The footprint of instantiations of the non-stationary fuzzy set from *Example 1*.

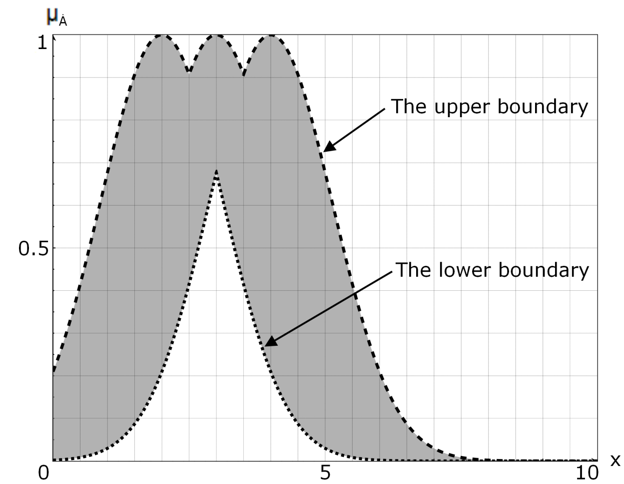

Fig. 3. The lower boundary, the upper boundary and the domain of instantiations of the non-stationary fuzzy set from *Example 1*.

where t ranges over T. An element of R_x is called the *temporal membership grade*.

Example 3: Continuing the example from above we get, for $x = 4$:

$$R_x = R_4 = \{A, B, C\},$$

and any of the A, B and C is a grade.

Definition 4: The *lower borderline* $\underline{\mu_{\dot{A}}}(x)$ is defined as:

$$\underline{\mu_{\dot{A}}}(x) = \bigcup_{x \in X} \{\inf(R_x)\}.$$

Definition 5: The *upper borderline* $\overline{\mu_{\dot{A}}}(x)$ is defined as:

$$\overline{\mu_{\dot{A}}}(x) = \bigcup_{x \in X} \{\sup(R_x)\}.$$

Definition 6: The *domain of instantiations* (DOI) is defined as:

$$\text{DOI} = \bigcup_{x \in X} [\underline{\mu_{\dot{A}}}(x), \overline{\mu_{\dot{A}}}(x)].$$

Definition 7: The *footprint of instantiations* (FOI) is defined as:

$$\text{FOI} = \bigcup_{(t,x) \in T \times X} \{\mu_{\dot{A}}(t, x)\} = \bigcup_{x \in X} R_x.$$

The footprint of instantiations of the non-stationary fuzzy set from *Example 1* is shown in Fig. 2. The lower and upper boundaries, and the domain of instantiations of the same set are shown in Fig. 3.

Definition 8: The *temporal histogram* is a function $h_x : R_x \to \mathbb{N}$ which associates with each element $\mu_{\dot{A}}(t, x) \in R_x$ the number of pairs in the temporal membership function graph G_x whose second element is $\mu_{\dot{A}}(t, x)$.

IV. Operations on Non-Stationary Fuzzy Sets

In this Section, the operators of *union*, *intersection* and *complement* of non-stationary fuzzy sets are introduced. To this end, we first recall the familiar properties of type-1 fuzzy sets. Suppose that we have two fuzzy sets, A and B, characterised by membership functions $\mu_A(x)$ and $\mu_B(x)$:

$$A = \int_{x \in X} \mu_A(x)/x,$$

$$B = \int_{x \in X} \mu_B(x)/x.$$

Recall that:

$$A \cup B = \int_{x \in X} \mu_{A \cup B}(x)/x,$$

$$A \cap B = \int_{x \in X} \mu_{A \cap B}(x)/x,$$

$$\bar{A} = \int_{x \in X} 1 - \mu_A(x)/x.$$

The membership functions of the union and intersection of A and B, and the complement of A are, of course:

$$\mu_{A \cup B}(x) = \mu_A(x) \oplus \mu_B(x), \ \forall x \in X,$$

where $\oplus$ is a t-conorm,

$$\mu_{A \cap B}(x) = \mu_A(x) \otimes \mu_B(x), \ \forall x \in X,$$

where $\otimes$ is a t-norm, and

$$\mu_{\bar{A}}(x) = \overline{\mu_A(x)}, \ \forall x \in X,$$

where $^{-}$ denotes a generic complement. Using the maximum t-conorm, the minimum t-norm and the standard complement, the above become:

$$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)], \ \forall x \in X,$$

$$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)], \ \forall x \in X,$$

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x),\ \forall x \in X.$$

Now, let $T = \{t_1, \ldots, t_n\}$ be a set of time points t_i, and let $\dot{A}$ and $\dot{B}$ be non-stationary fuzzy sets of a universe of discourse X. Thus:

$$\dot{A} = \int_{t \in T} \int_{x \in X} \mu_{\dot{A}}(t, x)/x/t$$

and

$$\dot{B} = \int_{t \in T} \int_{x \in X} \mu_{\dot{B}}(t, x)/x/t.$$

Definition 9: The union of $\dot{A}$ and $\dot{B}$ is a non-stationary fuzzy set $\dot{A} \cup \dot{B}$ such that:

$$\dot{A} \cup \dot{B} = \int_{t \in T} \int_{x \in X} \mu_{\dot{A} \cup \dot{B}}(t, x)/x/t,$$

where

$$\mu_{\dot{A} \cup \dot{B}}(t, x) = \mu_{\dot{A}}(t, x) \oplus \mu_{\dot{B}}(t, x),\ \forall (t, x) \in T \times X.$$

Using the maximum t-conorm, this becomes:

$$\mu_{\dot{A} \cup \dot{B}}(t, x) = \max[\mu_{\dot{A}}(t, x), \mu_{\dot{B}}(t, x)],\ \forall (t, x) \in T \times X.$$

Definition 10: The intersection of $\dot{A}$ and $\dot{B}$ is a non-stationary fuzzy set $\dot{A} \cap \dot{B}$ such that:

$$\dot{A} \cap \dot{B} = \int_{t \in T} \int_{x \in X} \mu_{\dot{A} \cap \dot{B}}(t, x)/x/t,$$

where

$$\mu_{\dot{A} \cap \dot{B}}(t, x) = \mu_{\dot{A}}(t, x) \otimes \mu_{\dot{B}}(t, x),\ \forall (t, x) \in T \times X.$$

Which, using the minimum t-norm, becomes:

$$\mu_{\dot{A} \cap \dot{B}}(t, x) = \min[\mu_{\dot{A}}(t, x), \mu_{\dot{B}}(t, x)],\ \forall (t, x) \in T \times X.$$

Definition 11: The complement of $\dot{A}$ is a non-stationary fuzzy set $\bar{\dot{A}}$ such that:

$$\bar{\dot{A}} = \int_{t \in T} \int_{x \in X} \mu_{\bar{\dot{A}}}(t, x)/x/t,$$

where

$$\mu_{\bar{\dot{A}}}(t, x) = \overline{\mu_{\dot{A}}(t, x)},\ \forall (t, x) \in T \times X.$$

Which, using the standard complement, becomes:

$$\mu_{\bar{\dot{A}}}(t, x) = 1 - \mu_{\dot{A}}(t, x),\ \forall (t, x) \in T \times X.$$

V. Selected Properties of Non-Stationary Fuzzy Set Operators

This section is dedicated to the proofs of fundamental properties of the non-stationary fuzzy set operators defined above. These proofs are derived directly from Zadeh's proofs for standard type-1 fuzzy sets; we include them for completeness. Table I summarises the set theoretic laws which are satisfied by non-stationary fuzzy sets.

Let us consider non-stationary fuzzy sets $\dot{A}$, $\dot{B}$ and $\dot{C}$:

$$\dot{A} = \int_{t \in T} \int_{x \in X} \mu_{\dot{A}}(t, x)/x/t,$$

$$\dot{B} = \int_{t \in T} \int_{x \in X} \mu_{\dot{B}}(t, x)/x/t$$

and

$$\dot{C} = \int_{t \in T} \int_{x \in X} \mu_{\dot{C}}(t, x)/x/t.$$

Note that, for the sake of brevity in the formulae below, whenever we use the non-stationary union, intersection or complement operators, defined in Section IV, we omit $\forall (t, x) \in T \times X$.

A. Involution

Let us consider the complement of $\dot{A}$, $\bar{\dot{A}}$:

$$\bar{\dot{A}} = \int_{t \in T} \int_{x \in X} \mu_{\bar{\dot{A}}}(t, x)/x/t.$$

By the definition of the standard complement operator for non-stationary fuzzy sets, we have:

$$\mu_{\bar{\dot{A}}}(t, x) = 1 - \mu_{\dot{A}}(t, x).$$

Thus, the complement of $\bar{\dot{A}}$ can be expressed as:

$$\bar{\bar{\dot{A}}} = \int_{t \in T} \int_{x \in X} \mu_{\bar{\bar{\dot{A}}}}(t, x)/x/t,$$

where

$$\mu_{\bar{\bar{\dot{A}}}}(t, x) = 1 - \mu_{\bar{\dot{A}}}(t, x).$$

Since $\mu_{\bar{\dot{A}}}(t, x) = 1 - \mu_{\dot{A}}(t, x)$, we obtain:

$$\mu_{\bar{\bar{\dot{A}}}}(t, x) = 1 - (1 - \mu_{\dot{A}}(t, x)).$$

It follows that:

$$\mu_{\bar{\bar{\dot{A}}}}(t, x) = \mu_{\dot{A}}(t, x).$$

B. Commutativity

1) Union: By the definition of the union operator for non-stationary fuzzy sets, we have:

$$\mu_{\dot{A} \cup \dot{B}}(t, x) = \mu_{\dot{A}}(t, x) \oplus \mu_{\dot{B}}(t, x).$$

As t-conorm operators are commutative, we know that:

$$\mu_{\dot{A}}(t, x) \oplus \mu_{\dot{B}}(t, x) = \mu_{\dot{B}}(t, x) \oplus \mu_{\dot{A}}(t, x).$$

Again, by the definition:

$$\mu_{\dot{B} \cup \dot{A}}(t, x) = \mu_{\dot{B}}(t, x) \oplus \mu_{\dot{A}}(t, x)$$

and so:

$$\mu_{\dot{A} \cup \dot{B}}(t, x) = \mu_{\dot{B} \cup \dot{A}}(t, x).$$

2) Intersection: The proof is similar to that above, substituting $\cap$ for $\cup$, t-norm for t-conorm and $\otimes$ for $\oplus$.

TABLE I

SELECTED SET THEORETIC LAWS SATISFIED BY NON-STATIONARY FUZZY SETS.

Involution	$\bar{\bar{\dot{A}}} = \dot{A}$	
	t-conorm	*t-norm*
Commutativity	$\dot{A} \cup \dot{B} = \dot{B} \cup \dot{A}$	$\dot{A} \cap \dot{B} = \dot{B} \cap \dot{A}$
Associativity	$\dot{A} \cup (\dot{B} \cup \dot{C}) = (\dot{A} \cup \dot{B}) \cup \dot{C}$	$\dot{A} \cap (\dot{B} \cap \dot{C}) = (\dot{A} \cap \dot{B}) \cap \dot{C}$
Distributivity	$\dot{A} \cap (\dot{B} \cup \dot{C}) = (\dot{A} \cap \dot{B}) \cup (\dot{A} \cap \dot{C})$	$\dot{A} \cup (\dot{B} \cap \dot{C}) = (\dot{A} \cup \dot{B}) \cap (\dot{A} \cup \dot{C})$
Idempotence	$\dot{A} \cup \dot{A} = \dot{A}$ for max only	$\dot{A} \cap \dot{A} = \dot{A}$ for min only

C. Associativity

1) Union: By the definition of the union operator for non-stationary fuzzy sets, we have:

$$\mu_{\dot{A} \cup (\dot{B} \cup \dot{C})}(t,x) = \mu_{\dot{A}}(t,x) \oplus (\mu_{\dot{B}}(t,x) \oplus \mu_{\dot{C}}(t,x)).$$

As t-conorm operators are associative, we know that:

$$\mu_{\dot{A}}(t,x) \oplus (\mu_{\dot{B}}(t,x) \oplus \mu_{\dot{C}}(t,x)) = \\ = (\mu_{\dot{A}}(t,x) \oplus \mu_{\dot{B}}(t,x)) \oplus \mu_{\dot{C}}(t,x).$$

Again, by the definition:

$$\mu_{(\dot{A} \cup \dot{B}) \cup \dot{C}}(t,x) = (\mu_{\dot{A}}(t,x) \oplus \mu_{\dot{B}}(t,x)) \oplus \mu_{\dot{C}}(t,x)$$

and so:

$$\mu_{\dot{A} \cup (\dot{B} \cup \dot{C})}(t,x) = \mu_{(\dot{A} \cup \dot{B}) \cup \dot{C}}(t,x).$$

2) Intersection: The proof is similar to that above, substituting $\cap$ for $\cup$, t-norm for t-conorm and $\otimes$ for $\oplus$.

D. Distributivity

The proofs of distributivity for non-stationary union and intersection follow exactly the form of those given above for associativity, and so were omitted for brevity.

E. Idempotence

1) Union: It is well known that only the maximum t-conorm is idempotent. Thus, by the definition of the union operator for non-stationary fuzzy sets, using the maximum t-conorm we obtain:

$$\mu_{\dot{A} \cup \dot{A}}(t,x) = \max(\mu_{\dot{A}}(t,x), \mu_{\dot{A}}(t,x)).$$

As the $\max$ operator is idempotent, we know that:

$$\max(\mu_{\dot{A}}(t,x), \mu_{\dot{A}}(t,x)) = \mu_{\dot{A}}(t,x)$$

and so:

$$\mu_{\dot{A} \cup \dot{A}}(t,x) = \mu_{\dot{A}}(t,x).$$

2) Intersection: As above, omitted for brevity.

VI. COMPARISON BETWEEN NON-STATIONARY AND TYPE-2 FUZZY SETS

Based on the concepts defined in Section III, we have drawn the first parallels between non-stationary and type-2 fuzzy sets. Some of the parallels are given in Table II. Clearly, the two kinds of sets are formally (on a mathematical level) different. Nevertheless, some informal resemblances still can be found, and similarities can be established on two levels.

On a graphical level. Equivalent concepts occupy the same position in figures depicting the sets. For example, the FOI is graphically equivalent to the FOU. Indeed, it can be seen that the FOI shown in Fig. 4 and the FOU shown in Fig. 5 resemble each other. However, on a formal level, the FOI and the FOU are different: the membership function $\mu_{\dot{A}}(t,x)$ is central to the idea of FOI, while $\mu_{\tilde{A}}(x,u)$ is not involved in the FOU. This is also evidenced by the labelling of axes in Figs. 4 and 5. As an aside, the DOI is visually even closer to the FOU. Mendel and John also introduced a concept known as the domain of uncertainty (DOU) for type-2 fuzzy sets [3], but the FOU is generally preferred.

On a correspondence level. Let P be a set, defined *a priori*, of the pairs of primitive concepts which, we claim, correspond to each other. Let $(a,b) \in P$, that is, a is equivalent on a correspondence level to b. Moreover, let $m(a)$ and $n(b)$ be new concepts obtained out of, respectively, a and b. Now, $m(a)$ corresponds to $n(b)$ if $n = m$. To clarify: assume a (a concept related to non-stationary fuzzy sets) is defined to correspond to b (a concept related to type-2 fuzzy sets). The operation m is now performed on a to obtain the new concept $m(a)$. If the operation n is performed on b, then the new concept $n(b)$ corresponds to $m(a)$ if the two operations are *the same*. For example, suppose we claim that the temporal membership function (of non-stationary sets) is equivalent to the secondary membership function (of type-2 sets). Now, T is the domain of the temporal membership function, while the primary membership of x is the domain of the secondary membership function [2]. Hence, T corresponds to the primary membership of x. On the other hand, if we claim that the range of the temporal membership function (the domain of the temporal histogram) is equivalent to the primary membership of x (the domain of the secondary membership function), then the temporal histogram corresponds to the secondary membership function.

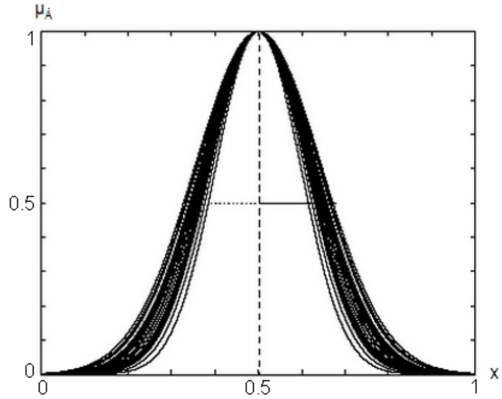

Fig. 4. The footprint of instantiations of a non-stationary fuzzy set.

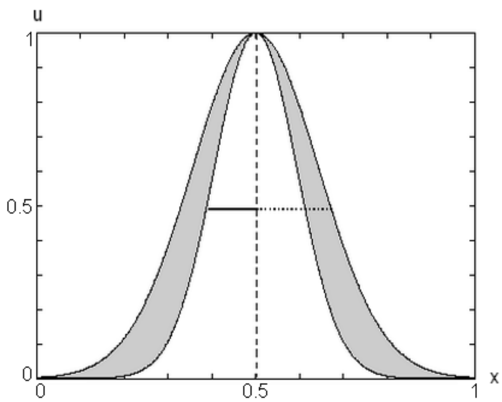

Fig. 5. The footprint of uncertainty of a type-2 fuzzy set.

From the above it is clear that a comparison of non-stationary and type-2 fuzzy sets is not trivial. We are now only beginning to investigate these issues, and the 'equivalence', as shown in Table II, is open to further discussion.

TABLE II
INFORMAL COMPARISON OF NON-STATIONARY AND TYPE-2 FUZZY SETS.

Non-stationary fuzzy sets	*Type-2 fuzzy sets*
Temporal membership function Normalized histogram	Secondary membership function
T	Primary membership of x
Temporal membership grade	Secondary membership grade
FOI	FOU

VII. CONCLUSIONS

We introduced non-stationary fuzzy sets as the means of capturing additional kinds of uncertainty compared to the standard type-1 and type-2 fuzzy sets. Specifically, we introduced them to mimic variability inherent in expert decision making, as illustrated in the case study given in [10]. In this paper, we provide a formal definition of non-stationary fuzzy sets and suggest a set of useful concepts relevant to them. We adapt the union, intersection and complement operators, introduced by Zadeh for type-1 fuzzy sets, to non-stationary fuzzy sets, and provide proofs of selected properties of these operations. Further, we begin the process of establishing more formal correspondence between non-stationary and type-2 fuzzy sets, and we can safely state that there is no simple equivalence between the two kinds of sets.

Since the concept of non-stationary fuzzy sets is novel, there are a lot of questions that may be addressed, e.g.:

- The extension of non-stationary fuzzy set operators to handle sets with a different number of time points.
- Comparison of non-stationary fuzzy sets and existing schemes characterizing both fuzziness and randomness (fuzzy random sets, fuzzy random variables, etc).
- The problem of defining the membership and perturbation functions in practical applications.

Each of the above issues constitutes an individual research topic beyond the scope of this paper and will be a subject of future work.

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